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187 - 192

$F° \text{ to } C° = C = \frac{5}{9}(F - 32)$

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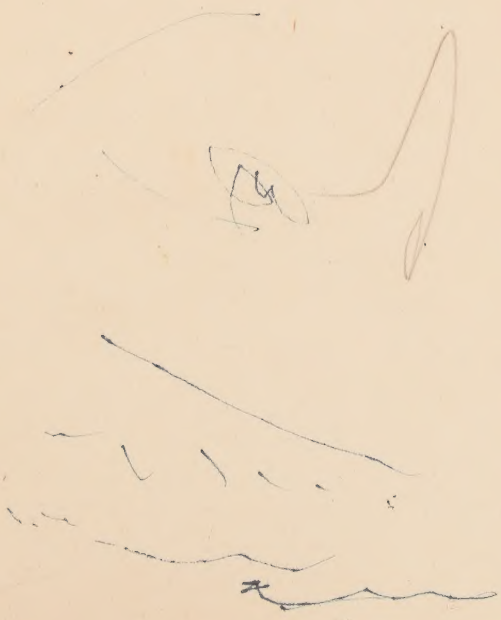

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SHOP MATHEMATICS

PART II
ADVANCED SHOP MATHEMATICS

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SHOP MATHEMATICS

PART II

ADVANCED SHOP MATHEMATICS

PREPARED IN THE
EXTENSION DIVISION OF
THE UNIVERSITY OF WISCONSIN

BY

EARLE B. NORRIS, M. E.
ASSOCIATE PROFESSOR OF MECHANICAL ENGINEERING,
IN CHARGE OF MECHANICAL ENGINEERING COURSES
IN THE UNIVERSITY EXTENSION DIVISION

AND

RALPH T. CRAIGO, B.S.
ASSISTANT PRINCIPAL, DUNWOODY INSTITUTE, MINNEAPOLIS
FORMERLY INSTRUCTOR IN MATHEMATICS, UNIVERSITY
EXTENSION DIVISION, UNIVERSITY OF WISCONSIN

McGRAW-HILL BOOK COMPANY, Inc.
NEW YORK AND LONDON

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PRINTED IN THE UNITED STATES OF AMERICA

XXIX

PREFACE

This volume presents the second half of the instruction papers in Shop Mathematics as developed and used in the Extension Division of the University of Wisconsin. In it the authors have endeavored to present such of the principles of algebra, geometry, trigonometry and logarithms as have been found to be of practical value in the shop, showing some of the better known applications and making the presentation as practical as seemed possible. Experience in the teaching of this course for over five years shows that it has been successful in developing in the student ability to apply the principles to his own shop problems and in giving him a good mathematical preparation for advanced technical study.

It is here presented in the belief that it will be found suitable for home study and for use as a text in trade schools, technical high schools and continuation schools.

The authors are indebted to Mr. Herbert J. Lehmann, Assistant in Applied Mathematics in the University Extension Division, for many valuable suggestions and criticisms; also to the authors and publishers of Van Velzer and Slichter's logarithm tables for permission to use their plates.

E. B. N.

The University of Wisconsin
Madison, Wisconsin,
July 1, 1913

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ADVANCED SHOP MATHEMATICS

CHAPTER I

FORMULAS

1. Value of Formulas.—All skilled workmen, such as machinists, pattern-makers, and electricians, must acquire and retain in their minds many rules which they must know when and how to use. These rules, when shortened by the use of letters and symbols to represent words, are called *formulas.*

Everyone is accustomed to using the ordinary signs of arithmetic: + for "plus"; × for "times"; = for "equals"; etc. The use of the Greek letter π to represent the number 3.1416 is also well known. Extending this practice, we find letters frequently used to represent words or figures, thereby shortening many statements and calculations.

A glance at any of the well-known trade magazines—the American Machinist, Power, Machinery, or the Metal Worker —or at any handbook, such as Kent's Mechanical Engineers' Handbook or The American Machinists' Handbook, will show how universal is the use of formulas and how essential to every good mechanic is a knowledge of formulas.

If a machinist wants to know the dimensions of a nut for a certain size of bolt or the size of drill for a tap, he can find the information in a formula; if the electrician wants to know the size of wire to carry a certain current a given distance with a certain drop in voltage, he uses a formula; the steam engineer also turns to a formula when he wishes to figure the horse-power of his engine; and likewise in all branches of engineering, formulas are in almost universal use.

Formulas applying to all branches of work are found in magazines and books; but to use them intelligently, to get the proper ones, and to fit them to the case in hand requires some little knowledge of the special mathematical principles that govern their formation and use.

2. The Use of Letters.—If we want to make a short, plain statement of the fact that "the circumference of a circle equals the number 3.1416 multiplied by the diameter of the circle," we write:

$$C = \pi \times D$$

where C = the circumference of the circle
D = the diameter of the circle, and
$\pi = 3.1416$

In this formula, π always stands for the constant number 3.1416; it is never used for any other quantity. D stands for a number or quantity that would be known for any particular problem we might be working but, until we apply the formula to some particular circle, it may represent any number. C stands for an unknown quantity, the value of which we are to find but which we will not know until we have worked the problem by multiplying 3.1416 by the diameter.

We see, therefore, that letters are used to represent three different kinds of quantities.

1. Constant quantities.
2. Quantities whose values are not the same for all problems but which become fixed as soon as a certain problem is to be worked by the formula.
3. Unknown quantities whose values we must calculate after the values of the other letters have been supplied.

The great advantage of formulas over the use of long sentences is that they show so clearly just what the relation is between quantities and what operations of arithmetic must be performed. It is nearly always necessary, however, in writing a formula to show in some way just what the letters stand for. This may be explained in words or by a sketch or drawing. For example, the formula for the area of a rectangle can be written

$$A = L \times B$$

We can then explain that

A is the area of a rectangle, of which
L is the length, and
B is the breadth.

Or we can explain it by merely drawing a sketch of a rectangle and marking on the rectangle what the letters mean, as in Fig. 1.

In most formulas, the letters used are the first letters of the

words they represent: D for diameter, C for circumference, A for area, L for length, B for breadth, H for height, etc. In such cases it is not always deemed necessary to explain what the letters stand for.

Sometimes there are several quantities having the same name. For example, a formula may involve the diameters of two circles. In such a case we can use the capital D and the small d. These are generally read *d major* and *d minor*.

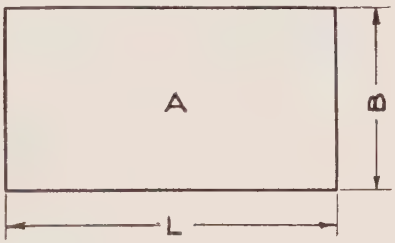


FIG. 1.

If there are several quantities of one kind, some other device must be used. In such a case we can distinguish the different letters by *primes* or *subscripts*. A' is *A prime*, A'' is *A second prime*, A''' is *A third prime*, etc. Likewise, A_1 is *A sub one*, A_2 is *A sub two*, A_3 is *A sub three*, etc. Remember that these primes and subscripts do not indicate any operations as do the exponents ^{2}, ^{3}, etc., but merely designate different quantities referred to by what would otherwise be the same letter.

3. The Omission of the Times Sign.—The rules of addition, subtraction, multiplication, division, and so forth, which govern work in arithmetic, apply in the same ways to problems in formulas. The first difference that is noticed is that the sign of multiplication, $\times$, is not needed and is seldom written. So far we have used this sign between letters, but it is more often left out. In the formula for the area of the rectangle we would write $A=LB$. The two letters LB written together signify that the numbers they represent are to be multiplied together.

In arithmetic, when two numbers are placed together, like 27, the first number indicates tens and the second, units. In formulas, when two letters are written together like LB, or a number and a letter, like $3D$, the process of multiplication is understood. If a rectangle is 7 in. by 3 in., the area is

$$A=LB$$
$$=7\times3=21 \text{ sq. in.}$$

The formula for the circumference of a circle is likewise generally written $C=\pi D$, the sign $\times$ being left out between π and D.

Occasionally two letters are used to represent a single quantity,

as *H. P.* for *horse-power*, or *M. A.* for *mechanical advantage*. In such a case, they should be treated as a single letter.

4. Substitution.—When we have a formula and wish to apply it to a particular problem, we *substitute* the given numbers for the corresponding letters in the formula. We can then perform the processes of arithmetic between the numbers and find the value of the remaining letter—the one that is left to be found after all the others have been replaced by numbers.

Example 1:

What is the circumference of a circle 12 in. in diameter?

$C = \pi D$
$C = 3.1416 \times 12$
$= 37.6992$ in., *Answer.*

Explanation: We know that π stands for 3.1416; according to the problem, D is 12 in., and we know that these are to be multiplied; since the letters are placed together without any sign. We substitute 3.1416 for π, and 12 for D, and place the sign $\times$ between them. This leaves $C = 3.1416 \times 12$ and, multiplying the numbers together, we find C is 37.6992. Since D was given in inches, the value obtained for C will be in inches.

Example 2:

What is the area of a circle 6 in. in diameter?

$A = .7854\ D^2$
$A = .7854 \times 6^2$
$= 28.2744$ sq. in., *Answer.*

Explanation: Here the example gives the value of the diameter, $D = 6$ in. As the formula indicates, this must be squared and then multiplied by .7854, giving 28.2744 sq. in. as the area.

Note.—This formula is quite often written, $A = \frac{\pi}{4} D^2$, because $.7854 = \frac{3.1416}{4}$, or $3.1416 \div 4$.

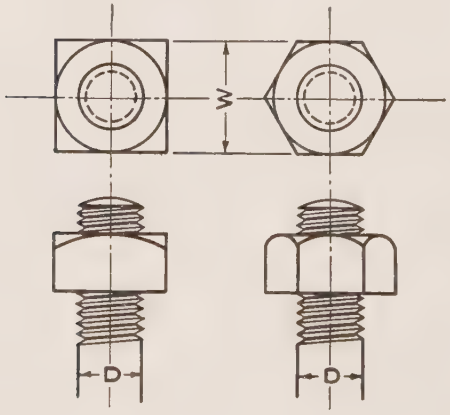


Fig. 2.

5. Order of Operations.—Sometimes one is puzzled to tell which operations in a formula should be performed first. For example, the formula for the width of a rough, U. S. standard nut or bolt-head in terms of the diameter of the bolt is

$$W = 1\tfrac{1}{2}\ D + \tfrac{1}{8} \text{ in.}$$

W and D being as shown in Fig. 2. This applies to either a square or a hexagonal nut. W is the distance between two opposite sides and is the opening of a wrench to grip

the nut. In using this formula the question comes up, "Shall I multiply the diameter by $1\frac{1}{2}$ and then add $\frac{1}{8}$ in., or shall I add $\frac{1}{8}$ in. to the diameter and then multiply the sum by $1\frac{1}{2}$?" There is a rule which applies to all such cases:

Unless otherwise indicated, operations of multiplication and division should be performed before those of addition and subtraction.

In the formula just given for the width of a nut or a bolt-head, this rule shows us that the $1\frac{1}{2}$ applies only to the D and that the $\frac{1}{8}$ in. is to be added afterward.

Example:

What is the width of a nut for a $\frac{7}{8}$-in. bolt?

$W = 1\frac{1}{2}\ D + \frac{1}{8}$ in.
$W = 1\frac{1}{2} \times \frac{7}{8} + \frac{1}{8}$ in.
$= \frac{3}{2} \times \frac{7}{8} + \frac{1}{8}$ in.
$= \frac{21}{16} + \frac{1}{8}$ in.
$= \frac{23}{16} = 1\frac{7}{16}$ in., *Answer.*

Explanation: The diameter being $\frac{7}{8}$ in. we substitute this for D. According to the rule, we first multiply the $\frac{7}{8}$ by the $1\frac{1}{2}$ and get $\frac{21}{16}$ as the product. To this we add the $\frac{1}{8}$ in., and obtain $1\frac{7}{16}$ in. as the width of the nut.

6. The Parenthesis.—Quite frequently there are cases where we want an operation of addition or subtraction to be done before one of multiplication or division. In such cases we must use some special means of indicating it so that no mistake can be made.

The most common method of doing this is by use of the *parenthesis* ( ).

For example, in calculating the area of a trapezoid, such as shown in Fig. 3, it is necessary to get the average width by taking half the sum of the upper and lower bases. This must then be multiplied by the height. As a formula, this would be written

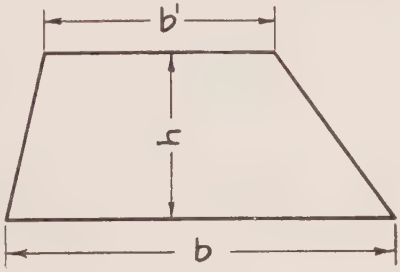


FIG. 3.

$$A = \tfrac{1}{2}(b + b')h$$

Here the use of the parenthesis around the $b + b'$ indicates that these two measurements are to be added first and their sum multiplied by $\frac{1}{2}$ and by h.

Example:

Find the area of a steel plate shaped like Fig. 3, with the lower base 6 ft., the upper base 4 ft., and the height 3 ft.

$$A = \frac{1}{2}(b+b')h$$
$$A = \frac{1}{2}\times(6+4)\times 3$$
$$= \frac{1}{2}\times 10\times 3$$
$$= 15 \text{ sq. ft., } \textit{Answer.}$$

Explanation: After substituting the values 6, 4, and 3 for b, b', and h, the parenthesis tells us that the 6 and 4 must be added first, giving 10. This 10 takes the place of the parenthesis and all that it contained. We then have a simple problem in multiplication to complete the work.

7. Writing Formulas.—In writing a formula for anything we proceed exactly as we would if we were working out an actual example in numbers, except that, where there are no numbers given, we put down letters to represent the missing dimensions or quantities. Many people prefer, even when working a numerical example, first to write out a formula in letters and then to substitute the given numbers. Frequently this will save time by showing how the work can be shortened.

Example:

Write the formula for the weight of a hollow castiron cylinder with outside diameter d_1, inside diameter d_2, and length l as shown in Fig. 4.

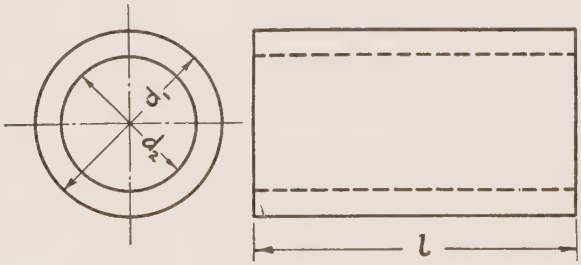


FIG. 4.

Explanation: In getting the weight we must first get the volume and multiply this by .26, the weight of a cubic inch of cast iron.

To get the volume we must first get the area on the end.

The area of the outer circle $= .7854\ d_1^2$

The area of the inner circle $= .7854\ d_2^2$

The net area on the end is

$$.7854\ d_1^2 - .7854\ d_2^2$$

Since both d_1^2 and d_2^2 are to be multiplied by .7854 we can shorten this to

$$.7854(d_1^2 - d_2^2)$$

This will give us only one multiplication by .7854 to perform instead of two.

Next we must multiply this by l to get the volume.

$$V = .7854(d_1^2 - d_2^2)l$$

And since 1 cu. in. of castiron weighs .26 lb., we can multiply this whole expression by .26 and get a formula for the weight.

$$W = .26\times .7854(d_1^2 - d_2^2)l$$

Note.—If very many of these cylinders were to be calculated, we could shorten the work further by multiplying $.26 \times .7854$ once for all. The formula would then read

$$W = .2042(d_1^2 - d_2^2)l$$

The formula assumes that the dimensions d_1, d_2, and l are all in inches. Hence care must be taken that these dimensions are all in inches when they are substituted in the formula.

PROBLEMS

1. From the formula for the area of a rectangle find the areas in the following cases:

(a) $L = 5$ in., $B = 3$ in.
(b) $L = 12$ ft., $B = 5$ ft.
(c) $L = 7\frac{1}{2}$ ft., $B = 3\frac{3}{4}$ ft.
(d) $L = 40$ yd., $B = 14$ yd.

2. Find the widths of the rough, U. S. standard nuts for bolts of the following diameters: $\frac{5}{16}$ in., $\frac{1}{2}$ in., $\frac{3}{4}$ in., $1\frac{1}{8}$ in., and $1\frac{1}{2}$ in.

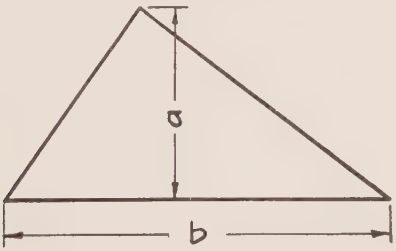


Fig. 5.

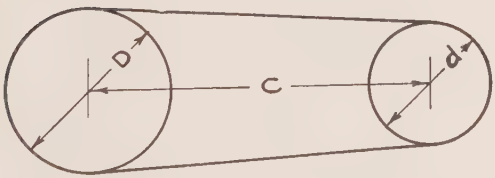


Fig. 6.

3. The area of a triangle is one-half the product of the base by the altitude. Using the dimensions shown in Fig. 5, write a formula for the area and from it find the areas of the following triangles:

(a) $a = 5$ in., $b = 6$ in.
(b) $a = 12$ in., $b = 4\frac{1}{2}$ in.
(c) $a = 11$ in., $b = 3\frac{1}{4}$ in.
(d) $a = 7\frac{1}{2}$ ft., $b = 13$ ft.

4. To get the approximate length of belt to order for two pulleys as shown in Fig. 6, we can develop a formula as follows: There are two straight lengths each a little greater than the distance C between shafts. Around

the larger pulley is a length somewhat greater than $\frac{\pi}{2} D$; and around the smaller pulley a length somewhat less than $\frac{\pi}{2} d$. Altogether, the length is a little more than $\pi \frac{(D+d)}{2} + 2\ C$. To be on the safe side we can replace π by 3.25. This will allow a little waste to be trimmed off when the belt is put in place and laced. The formula for the length to order is

$$L = 3.25 \frac{(D+d)}{2} + 2\ C$$

Find the length of belt for 36-in. and 24-in. pulleys on shafts 16 ft. apart.

5. The length of stock needed to make a welded ring as shown in Fig. 7 is equal to the length of the dotted circle midway between the inside and outside of the ring. The diameter of this circle is $D+T$. The circumference of this circle is $\pi(D+T)$. For this work it is sufficiently accurate and much simpler to use $3\frac{1}{7}$ or $\frac{22}{7}$ for π. Hence,

$$L = \frac{22}{7} (D+T)$$

Calculate the length of bar $\frac{1}{2}$ in. thick needed to make a ring of 10 in. inside diameter.

Fig. 7.

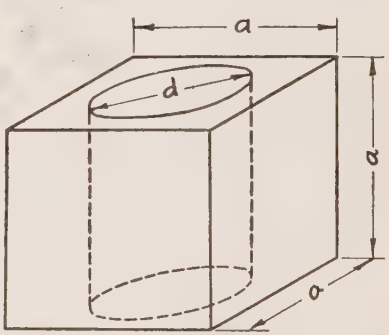


Fig. 8.

6. What would be the formula for the *volume* of a flat rectangular plate of uniform thickness?

7. What would be the formula for the *weight* of this plate if made of steel? (Weight of steel = .283 lb. per cu. in.)

8. Write out the formula and calculate the weight of a circular steel boiler head 64 in. in diameter and $\frac{7}{16}$ in. thick.

9. A good rule for close running fits is to have the shaft smaller than the hole by .003 in. + an additional one-thousandth for each inch of diameter. In other words, if d represents the size of the hole, then

$$\text{Size of shaft} = d - (.003 + .001\ d)$$

According to this, to what size would you grind a shaft to give it a close running fit in a 2-in. hole?

10. Write a formula for the weight of the cast iron block shown in Fig. 8, the piece being a cube having a length a on each edge and having a hole of diameter d drilled through it.

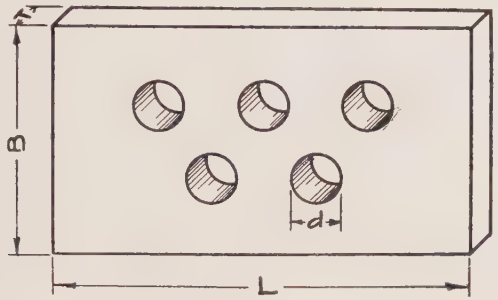


Fig. 9.

11. Fig. 9 represents a steel plate of dimensions B, L, and T, with 5 holes of diameter d punched in it. Write out a formula for the volume of the plate.

12. Suppose that in Fig. 9, there had been, instead of 5 holes, any number of holes represented by n. Write out a formula for the volume of such a plate.

CHAPTER II

EXPLANATION OF TERMS: ALGEBRAIC ADDITION

8. Expressions and Terms.—Any collection of letters and numbers that represents some quantity and that might, by substitution, be reduced to a single number is called an *expression*. For example: πr^2, $d_1^2 - d_2^2$, $a+b+c$, and $1\frac{1}{2}D+\frac{1}{8}$ *in.* are expressions.

An expression not divided into parts by a + or − sign is called a *term*. An expression may contain one term or several terms. The expression πr^2 is a single term; $d_1^2 - d_2^2$ contains two terms, d_1^2 and $-d_2^2$; $a+b+c$ has three terms; while $1\frac{1}{2}D+\frac{1}{8}$ in. has two terms.

A term is built up of *coefficients*, *bases*, and *exponents*. In the term $3a^2$, 3 is the coefficient, a is the base, and 2 is the exponent. The coefficient is the number part of the term and indicates how many times the quantity expressed by the letters is to be taken. Thus $3a^2$ means a^2 taken three times, or $3 \times a^2$, or $a^2+a^2+a^2$. If a represents a certain length or distance, as

FIG. 10.

shown at the left in Fig. 10, then a^2 will represent the area of a square figure having a length a on each side. The term $3a^2$ will represent an area made up of 3 of these squares.

If a cube has each edge x inches long, then the area of each face is x^2 sq. in. Since there are six faces, the total surface of the block is $6x^2$.

If a term has no coefficient, it is understood that the coefficient is 1. Thus L means $1L$, or $1 \times L$; D^2 means $1D^2$, or $1 \times D^2$.

9. Like and Unlike Terms.—Terms that have the same bases with the same exponents, and that differ only in the coefficient or numerical part, are called *like terms*. Thus, $3a^2$ and $6a^2$ are

like terms, since they evidently refer to the same things but differ only in the quantity of things. If a represents inches, then a^2 will represent *square* inches; 3 a^2 and 6 a^2 will both represent square inches but in different amounts. From this it will be seen that like terms are added by adding the coefficients; the sum of 3 a^2 and 6 a^2 is 9 a^2, just as the sum of 3 *sq. in.* and 6 *sq. in.* is 9 *sq. in.*, or as 3 *nuts*+6 *nuts*=9 *nuts*. Likewise, like terms are subtracted by subtracting coefficients, 5 $D^2-D^2=4\ D^2$.

Terms must contain the same letters and the letters must have the same exponents if they are to be added in this way; in other words, they must be *like* terms. For example, 3 ab and 4 ab^2 cannot be added to form a single term, because they are not like

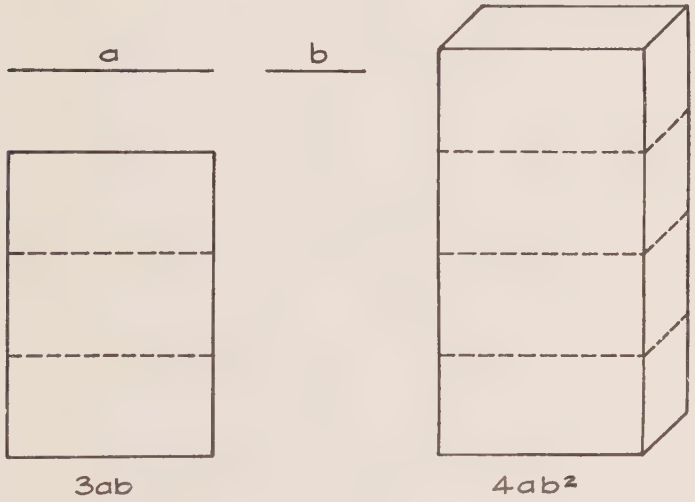


Fig. 11.

terms. The coefficients do not represent quantities of the same kind of things. These can no more be added to give a single term than can 3 nuts and 4 washers. They must still remain nuts and washers; we cannot call them all nuts or all washers. We can merely express the addition by writing 3 *nuts*+4 *washers*. Fig. 11 shows that if a and b represent two different lengths of lines, then 3 ab represents a certain area in square inches while 4 ab^2 represents a certain volume in cubic inches. They are therefore entirely different kinds of things. Hence, in adding 3 ab and 4 ab^2, we can merely express the addition, 3 ab+4 ab^2, but cannot combine them into a single term.

10. Positive and Negative Quantities.—In working examples in arithmetic, we consider a + or − sign between two quantities as representing an operation to be performed. We can do the same in the study of formulas but, since these operations cannot always be actually performed, it is sometimes necessary to look at these signs in a little different way. The best plan is to consider a sign as belonging to or being a part of the quantity that follows it.

Fig. 12.

Quantities or terms preceded by a + sign are called *positive quantities*, while quantities preceded by a − sign are *negative quantities*.

We often see temperatures above zero indicated by the + sign and temperatures below zero by the − sign. Thus + 60° means 60° above zero, while − 30° means 30° below zero. The signs + and − are here seen to indicate a direction—*above* or *below* zero.

In keeping the score of a game, if a man is "set back" or loses a certain number of points, he scores a minus or a negative quantity. If he has lost in this way more than he has won, then his total score will be a negative quantity, or, as we sometimes say, he is "in the hole."

In general, quantities that mean a gain or increase are positive; quantities that mean a loss or decrease are negative.

If a thermometer stands at +70° and then changes −15°, the mercury will then stand at +70°−15°= +55°. If, on another day, the thermometer stands at +10° and then changes −30°, the mercury will then stand at +10°−30°= −20°. See Fig. 12.

If a man overdraws his bank account, his balance at the bank will be a negative quantity, meaning that he owes the bank instead of the bank's owing him. If he has a balance of $7.50 but draws a check for $10.00, his balance will be −$2.50. If, later, he deposits $17.50, his balance will then be −$2.50+$17.50= +$15.00

$$7.50-10.00= -2.50$$
$$-2.50+17.50= +15.00$$

When a term has no sign before it, it is understood that it is a positive quantity. It will be noticed that the first term in

an equation or expression seldom has a sign before it. In such cases it is understood that that term is positive. Thus, $a-b+c$ means $+a-b+c$.

11. Algebraic Sums.—When a number of like terms are connected by + and − signs, they can be combined into a single term. In doing this we add the coefficients of the + terms and then subtract the coefficients of the −terms. This is called *finding the algebraic sum*, meaning that we take account of the signs and add or subtract according as the signs are + or −. To get the algebraic sum of $3\ ab-ab+7\ ab-4\ ab$ we find the algebraic sum of the coefficients and place it before the common base. The coefficients are $+3-1+7-4$. The positive coefficients are $+3$ and $+7$, making a total of $+10$; the negative coefficients are -1 and -4, or a total of -5. The combined value of all is $+10-5=+5$. Hence the sum of the terms $3\ ab-ab+7\ ab-4\ ab$ is $+5\ ab$ or just $5\ ab$, since it is not necessary to write the + when the term stands alone, the sign being understood.

$$3\ ab-ab+7\ ab-4\ ab=5\ ab$$

It must be remembered that only terms having the same letters with the same exponents can be actually added in this way. In the expression $8\ x+3\ y-5\ x+4\ y-2\ x-3\ y$, we can get the algebraic sum of the quantities containing x and also the algebraic sum of the quantities containing y; we can then express the addition of the resulting x and y terms. $8\ x-5\ x-2\ x=+1\ x$, or x. $3\ y+4\ y-3\ y=+4\ y$. Hence the entire expression can be reduced to $x+4\ y$.

$$8\ x+3\ y-5\ x+4\ y-2\ x-3\ y=x+4\ y$$

12. Addition.—In getting the algebraic sums of like terms we must understand the effect of the positive and negative signs.

If both terms to be added have positive signs, the addition is performed just as in arithmetic and is very simple.

$$+5\ D^2+4\ D^2=+9\ D^2$$

Here the terms are both positive and their sum is obtained by adding the coefficients 5 and 4 and writing the sum +9 with the D^2 attached.

If both terms to be added have negative signs, the coefficients are likewise added and a negative or minus sign is placed before the result.

$$-10\ a+-5\ a=-15\ a$$

This can be likened to the thermometer being 10 degrees below zero (-10) and then going down 5 degrees more (-5); the resulting temperature will be 15 degrees below zero (-15).

If one of the terms to be added is positive and the other negative, the coefficient of the result is the difference between the two coefficients; the sign of the result will be the sign of the larger of the two coefficients.

$$\begin{aligned} 26\,X - 16\,X &= 10\,X \\ -15\,M + 4\,M &= -11\,M \\ 3\,D^3 - 10\,D^3 &= -\,7\,D^3 \end{aligned}$$

In the first of these, the problem is clearly one of simple subtraction, the result being positive because the larger coefficient (26) is positive. To show the reasoning for the second case let us again liken it to a thermometer. If the mercury is 15 degrees below zero (-15) and then goes up 4 degrees ($+4$), it will then stand at 11 degrees below zero (-11). The third case can be explained in the same way. If the mercury is 3 degrees above zero ($+3$) and then goes down 10 degrees (-10), it will then stand at 7 degrees below zero (-7).

If one of the terms is positive and the other negative and their coefficients are the same, their sum is zero (0).

$$2\,c - 2\,c = 0$$

13. Addition of Polynomials.—When an expression contains two or more terms it is called a *polynomial*, meaning *many numbers*. Sometimes, especially in solving problems by formulas or algebra, it is necessary to add expressions containing more than one term each. In such cases we can only really add the like terms, that is, those that contain the same letters with the same exponents. The addition between unlike terms we can only express by the signs + or −.

Example 1:

Find the sum of $3\,X^2 + 2\,X + 1$ and $X^2 - 4\,X + 4$.

$3\,X^2 + 2\,X + 1$
$\,X^2 - 4\,X + 4$
$4\,X^2 - 2\,X + 5$, *Answer.*

Explanation: We first arrange the expressions with the like terms placed in vertical columns so they can be added conveniently. We can start to add either at the right or the left. Starting at the left we have the two like terms $3\,X^2$ and X^2. These are both positive and their sum is $+3\,X^2 + 1\,X^2 = +4\,X^2$ Since the $4\,X^2$ is the first term of the sum, it is not necessary to write the + before it. Next we have $+2\,X$ and $-4\,X$. Their algebraic sum is $-2\,X$. We write this below the line and then proceed to the next terms. Here we have $+1$ and $+4$ and their sum is $+5$, which we also write

below the line. The sum of the two expressions is, therefore, $4X^2-2X+5$. These are all unlike terms and consequently cannot be added any more than to express the addition by the signs.

Example 2:

Add D^2+1, $D-3$, and D^2+D+2.

$$\begin{array}{r} D^2 \quad +1 \\ \quad D-3 \\ D^2+ \quad D+2 \\ \hline 2\,D^2+2\,D \end{array}$$, *Answer.*

Explanation: In arranging these quantities, we find that there are three different kinds of terms: those containing D^2, those containing D, and those containing only numbers. Adding D^2 and D^2 we get $2\,D^2$. Likewise, D and D give $2\,D$. In adding the numbers, we have $+1$ and $+2$ which gives $+3$ and this with the -3 leaves zero (0), which it is not necessary to write in the sum. Hence the sum of the three quantities is $2\,D^2+2\,D$.

PROBLEMS

13. Find the sum of 3 bolts + 6 bolts + 12 bolts.

14. Find the sum of 7 bolts + 5 bolts + 5 nuts + 7 nuts + 24 washers.

15. What is 27 degrees − 15 degrees?

16. What is 12 degrees − 17 degrees?

17. How much is −5 dollars + 15 dollars?

18. If a man has property worth \$6000 but owes \$3000 to one man and \$5000 to another, how much money is he worth?

19. Find the sum of $4-3+6$

20. Find the sum of $6X-3X-5X$.

21. Add $3D+2$ and $D-0.5$.

22. Add $a+b+c$ and $a+b-c$.

23. Add $2D^2-D-6$ and $-D^2+3D+4$.

24. A foundry foreman keeps a record of the number of castings to be made from a certain pattern by putting down the number with a + sign whenever he receives an order for castings from this pattern, and by writing with a − sign the number of the castings made each day. How many castings are still to be made if his record appears as follows: $800-64-75-68-132+200-130-72-128$.

25. The capacity of a barrel in U. S. gallons is approximately

$$G=.0034\,h\left(\frac{d_1+d_2}{2}\right)^2$$

where h is the distance between the heads, inside, and d_1 and d_2 are the inside diameters at the head and bung respectively. What would be the capacity, in gallons, of a barrel having $h=27$, $d_1=17$, and $d_2=20$ in.?

26. A shipment of forgings contains 80 forgings weighing a pounds each and 64 forgings weighing b pounds each; a second shipment contains 50 of those weighing a pounds and 75 of those weighing b pounds. Out of these, 8 of those weighing a pounds and 12 of those weighing b pounds are returned as defective. Calculate the total weight of good forgings delivered, using + for shipped and − for returned.

CHAPTER III

ALGEBRAIC SUBTRACTION

14. Subtraction of Like Terms.—The same principles that govern the subtraction of denominate numbers in arithmetic apply to the subtraction of algebraic terms. The terms must represent the same kind of things; that is, they must be *like* terms if we are to actually perform the operation of subtraction. We can subtract 6 *ft.* from 12 *ft.* or 6 *bu.* from 12 *bu.*, but we cannot subtract 6 *ft.* from 12 *bu.* Neither can we subtract 6 x from 12 b, nor 3 a^2 from 5 a, because they are not the same kind of things. The following examples show the subtraction of like terms:

16 *in.*	12 x	10 a^3	10 *cu. ft.*	$2\,a^2c$
7 *in.*	4 x	4 a^3	4 *cu. ft.*	$5\,a^2c$
9 *in.*	8 x	6 a^3	6 *cu. ft.*	$-\,3a^2c$

In cases where the difference between *unlike* terms is asked for, we cannot perform the actual operation but can merely indicate the subtraction. Thus the difference between x^2 and y^2 can be indicated by the expression x^2-y^2.

Because terms are unlike in appearance does not mean that they can never be subtracted. Possibly, if we knew the values of the letters, they might turn out to be the same kind of things; the terms could then be reduced to like terms and the subtraction could be performed. In the case of x^2-y^2, so long as these terms stand as they are they cannot be subtracted but, possibly, if we knew the values of x and y, these values might be substituted for x and y and then, if the terms turned out to represent the same kind of things, the subtraction might be completed.

15. Subtraction of Negative Quantities.—Quite often subtraction is complicated by the fact that one or both of the terms are negative. For example we may want the difference between $3\,a$ and $-3\,a$. The performing of such subtractions is much simplified if we remember that subtraction means finding *the difference between* two quantities.

To get a clear idea of how to treat such problems, let us again take the illustration of the thermometer. Remember that degrees above zero are + and degrees below zero are −; also that changes in temperature upward are + and changes downward are −. When we find the difference between two points on the thermometer, we find the amount that the temperature must change to go from the one temperature to the other. If the mercury must *drop* to make the change, the difference of the two quantities should come out *negative* (−); if the mercury must *rise*, the difference should come out *positive* (+). Accordingly, if we subtract $-3°$ from $+17°$ we should get $+20°$ for the difference, because the mercury must rise 20° (+20°) to go from $-3°$ to $+17°$.

In a similar manner, it will be seen that if we subtract $+3°$ from $-17°$ we should get $-20°$, because the mercury would have to descend 20° (−20°) to go from $+3°$ to $-17°$. By the same process of reasoning we can explain the following examples of subtraction.

$20° - -10° = (+)30°$	$-6° - -15° = (+)9°$
$20x - -10x = 30x$	$-6a - -15a = 9a$
$6a^2 - -3a^2 = 9a^2$	$-4D^2 - -7D^2 = 3D$

From these examples it will be seen that *subtracting a negative quantity is the same as adding a positive quantity.* In other words, *two minus signs make a plus sign.* On the thermometer we can see that taking away negative degrees, or degrees of cold, would mean making it hotter, or adding + degrees.

The principle here brought out should be followed in working all problems in subtraction where negative terms are involved, as it saves much confusion and will aid in avoiding mistakes. It can be expressed in the following simple rule that will apply to all problems in subtraction of algebraic terms.

Rule.—*In subtracting algebraic quantities, change the signs of the subtrahend (change + to − and − to +) and then add the quantities.*

Example 1:

From $4D$ subtract $-3D$.

$4D$
$-3D$
$7D$, *Answer.*

Explanation: Subtracting $-3D$ is the same as adding $+3D$. In other words, we (mentally) change the sign of the subtrahend ($-3D$) and add, getting $4D + 3D = 7D$.

Example 2:

From $-21\ X^2$ take $7\ X^2$

$-21\ X^2$
$7X^2$
$-28\ X^2$, *Answer.*

Explanation: Subtracting $(+)\ 7\ X^2$ is the same as adding $-7\ X^2$; and $-21\ X^2+-7\ X^2=-28\ X^2$

16. Subtraction of Polynomials.—In subtracting polynomials the same process is followed. However, we must remember that we can only really subtract like terms; if the terms are unlike, the process can only be indicated.

Example 1:

Take $3\ X-12$ from $X^2-2\ X+1$.

$X^2-2\ X+1$
$3\ X-12$
$X^2-5\ X+13$, *Answer.*

Explanation: We must first arrange the expressions so that the terms in the subtrahend are placed under their like terms in the minuend. First, there is no term to be taken from the X^2, so that will be in the remainder. Next, subtracting $3\ X$ from $-2\ X$ is the same as adding $-3\ X$; and $-2\ X+-3\ X=-5\ X$. Then, subtracting -12 from $+1$ is the same as adding $+12$ to $+1$ and gives $+13$.

Example 2:

Subtract $a-b+c$ from $a+b-c$.

$a+b-c$
$a-b+c$
$2b-2c$, *Answer.*

Explanation: Subtracting a from a leaves zero, so we leave this space blank in the answer. Taking $-b$ from b is the same as adding $+b$ to $+b$ and gives $(+)\ 2\ b$. Likewise taking $+c$ from $-c$ is the same as adding $-c$ and $-c$ and gives $-2\ c$. Hence, the result is $2\ b-2\ c$.

Example 3:

Subtract $a^2-2\ ab+b^2$ from $a^2+2\ ab+b^2$.

$a^2+2\ ab+b^2$
$a^2-2\ ab+b^2$
$4\ ab$, *Answer.*

Explanation: Subtracting a^2 from a^2 leaves zero and, likewise, b^2 from b^2 leaves zero. Changing the $-2\ ab$ to $+2\ ab$ and then adding it to the $+2\ ab$ in the minuend gives $+4\ ab$.

Note.—In subtracting these two expressions we are finding the *difference* between them. There is no difference except as regards the second terms and here $+2\ ab$ is very clearly $4\ ab$ more than $-2\ ab$, just as having \$2 is \$4 better than being \$2 in debt.

17. Signs of Aggregation.—The use of the parenthesis has already been explained. There are other signs that are used for the same purpose of indicating that whatever is contained within them is to be considered as a single quantity.

It is often necessary to work problems like the following: "How much is the sum of $8+5$ and $8-5$?" This means that

first 8 and 5 are to be added, giving 13; then the difference between 8 and 5 is to be found, which is 3; then the sum (13) is to be added to the difference (3), equalling 16. This would be written

$$(8+5)+(8-5)=13+3=16.$$

Parentheses, ( ); braces, { }; brackets [ ]; and bars, $\overline{\quad}$, are used to indicate that all the terms contained within the signs are to be considered together. These signs are, therefore, called *signs of aggregation.*

(10−5), or {10−5}, or 10−5 means that the inclosed two terms are to be considered as a single term and that the operation indicated on the inside is to be performed first and that any operation indicated on the outside applies to the entire quantity within the sign of aggregation.

$$\begin{aligned}(10+6)-\{18-6\}-\overline{3-2}&=\\ (16)-\ \{12\}\ -1\ &=\\ 16-\ \ 12\ \ -1\ &=3.\end{aligned}$$

A sign of addition, subtraction, multiplication, or division placed before a parenthesis, or similar sign, refers to the entire expression included within this sign. It should also be remembered that if the first term within a parenthesis has no sign immediately before it, the plus sign is understood and the term is positive. For example {18−6} means {+18−6}. If there were a − sign before the expression, as −{18−6}, it would mean −{+18−6}, the − sign outside referring to the entire quantity within the braces, namely, +18−6. Therefore, −{18−6} = −{+18−6} = −{+12} = −12. Especial care must be taken when the bar is used. $-18-\overline{6}$ does not mean −18−6 but −(+18−6). That is, a sign immediately preceding and not under the bar refers to the entire quantity under the bar and *not* to the first term under the bar.

In constructing formulas or equations, we often get a number of such expressions and it is desirable to simplify them by getting rid of these signs if possible.

Whenever possible, first combine the terms inside the parentheses or braces, or under the bar. If one or more signs of aggregation are enclosed within others, always combine the inside ones first.

Example:

Simplify $18-\{10+(8-\overline{12-6})\}$.

$18-\{10+(8-\overline{12-6})\}$
$=18-\{10+(8-6)\}$
$=18-\{10+2\}$
$=18-12$
$=6$, *Answer.*

Explanation: The inside expression is that under the bar, $12-6$, and this is reduced to 6, the $-$ sign being put before it because it was before the $12-6$. Next, we combine the quantities in the parenthesis, getting 2. Now we can combine the terms in the bracket; and finally we can subtract this result from 18, giving 6 as the result of the whole expression.

Signs of aggregation preceded by a plus sign may be removed without changing the value of the expression.

For example: $(8+5)+(8-5)$ is the same as $8+5+8-5=16$.

Signs of aggregation preceded by a minus sign may be removed by reversing all of the inclosed signs.

$$(8+5)-(8-5)=8+5-8+5=10$$

Notice that the last parenthesis was preceded by a minus sign; hence, when the parenthesis was dropped it was necessary to change the enclosed signs, the $+8$ being made -8 and the -5 being changed to $+5$.

When the terms of an expression are not like terms, so they cannot be combined, it is best to follow these rules in order to simplify the expression.

Example:

$8-\{X-(5+\overline{X+6})\}=?$

$8-\{X-(5+\overline{X+6})\}$
$=8-\{X-(5+X+6)\}$
$=8-\{X-5-X-6\}$
$=8-X+5+X+6$
$=8+5+6-X+X$
$=19$, *Answer.*

Explanation: First we can remove the bar without any change, because it was preceded by a $+$ sign. The parenthesis has a $-$ sign before it. To remove this we change all the signs of the terms inside. Next we drop the braces in the same way. We now have the numbers $8+5+6$, whose sum is 19, and the X terms, $-X$ and $+X$. The sum of $-X$ and $+X$ is zero (0), so they cancel, leaving 19 as the value of the whole expression.

PROBLEMS

27. Subtract: 6 *ft.* from 12 *ft.*; 6 X^2 from 12 X^2; 7 *lb.* from 23 *lb.*; 7 *D* from 23 *D*.

28. Subtract: $-15°$ from $45°$; -20 *m* from 30 *m*; $-\$6$ from \$12; -18 *A* from 12 *A*.

29. Find the differences in the following problems:

\$ 4	6 *X*	6 *D*	$-3\ a^2$	5 *lb.*	$-7\ D^2$	$-\$\ 5$
\$10	3 *X*	12 *D*	$-12\ a^2$	-13 *lb.*	22 D^2	$-\$15$

30. Subtract:

6 *ft.* 4 *in.* from 10 *ft.* 6 *in.*;
$6a+4b$ from $10a+6b$;
$3x+2y$ from $4x-2y$;
$-2m-3n$ from $3m-7n$;

31. Subtract:

$3a-2$ from a^2-2a+1;
$4X+6$ from X^2+6;
$2D-1$ from D^2;
a^2-b^2 from $a^2+2ab+b^2$.

32. A *finished* nut has a width $\frac{1}{16}$ in. less than that of the rough nut as given in Article 5, Chapter I. What would be the formula for the width of a finished nut in terms of the bolt diameter?

33. Simplify the expression

$$(3\tfrac{1}{7}D+2)-(D+6)+3(D-1)$$

Simplify the expression

$$\{3(x-7)-\overline{2x+3}\}-[(x+2)-4(x+3)]$$

34. A formula often used for finding the areas of triangles when the lengths of the three sides are given, is

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

in which a, b, and c are the three sides, as shown in Fig. 13, and s is one-half the sum of the three sides.

From this formula find the area of the floor of a triangular warehouse whose sides are 68, 60, and 105 ft. long.

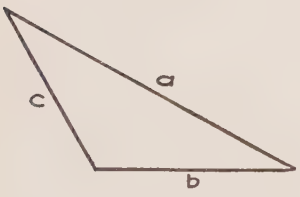


Fig. 13.

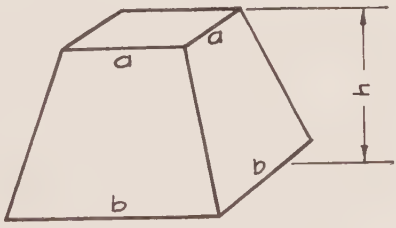


Fig. 14.

35. Another formula for the area of a triangle is

$$A=\frac{1}{4}\sqrt{[(a+b)^2-c^2]\,[c^2-(a-b)^2]}$$

Using this formula, calculate the floor area of the same warehouse as in problem 34.

36. Fig. 14 shows a frustum of a pyramid with the lower base b in. square, the upper base a in. square, and a perpendicular height of h in. The formula for the area of all 6 faces is

$$A=a^2+b^2+4\left(\frac{a+b}{2}\right)\sqrt{\left(\frac{b-a}{2}\right)^2+h^2}$$

How many square inches of sheet copper would be used in covering such a block if $a = 4$ in., $b = 6$ in., and $h = 5$ in.?

37. What would be the volume, in cu. in., of the same block if it can be calculated by the formula

$$V = (a^2 + b^2 + ab)\frac{h}{3}$$

CHAPTER IV

TRANSFORMATION OF FORMULAS

18. Equations.—An equation is a statement of equality between two quantities or expressions. An equation, therefore, consists of two expressions on opposite sides of an equality sign (=). For example, $W = 1\frac{1}{2}\ D + \frac{1}{8}$ in. is an equation. $C = \pi\ D$ is another equation. In each of these, the expression on the right of the = sign is identical in value with the expression on the left, and the statement of this equality forms an equation.

The two sides of an equation must always be equal or balance, just as the weights in the two pans of the druggists scales in Fig. 15 must be equal if the scales are to balance.

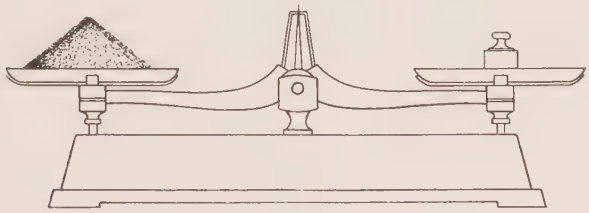


FIG. 15.

19. Transformations.—Formulas are generally written with a single letter on the left, representing the quantity whose value is usually to be found. The letters representing the known quantities are arranged on the right-hand side of the equation as a formula for this unknown quantity.

The formula for the area of a circle, $A = .7854\ D^2$, is written this way because in *most* cases it is used to get the value of the area when the diameter is known. If the area is known and the diameter is sought, the formula must be changed in form to show how the value of the diameter would be obtained from the value given for the area. In other words, we would want the formula changed into the form

$$D = \sqrt{\frac{A}{.7854}}$$

The horse-power formula is usually written

$$H = \frac{P\,L\,A\,N}{33000}$$

If, however, we want to find how many strokes per minute (N) an engine must make to give a certain horse-power, it would be more convenient to have this formula in a form having N on the left-hand side and all the other letters on the right. The process of changing formulas in this way is called *transformation*, or *solving the equation* for the desired letter. The process of doing this depends on one or two very simple but important laws.

It must be borne in mind that, whatever changes or transformations are made in a formula, the equality between the two sides of the equation must not be destroyed. No matter to what form the formula is changed, the expression which we have on the left-hand side must be equal to the expression on the right.

20. Transposition.—The process of changing a letter or number from one side of an equation to the other without disturbing the equality is called *transposing* or *transposition.*

If a steel bar 10 in. long has a mark placed so as to divide it into two parts 7 in. and 3 in. long, as shown in Fig. 16, the length of the bar can be expressed by 10 in. or by 7 in.+3 in.

Then, $$10 \text{ in.} = 3 \text{ in.} + 7 \text{ in.}$$

It is evident that one part is equal to the whole length minus the other part.

$$10 \text{ in.} - 7 \text{ in.} = 3 \text{ in.}$$

In this last equation the equality is still retained, but we have transposed one term from the right of the original equation to the left. Notice that the 7 in. has been transposed by changing its sign from + to −.

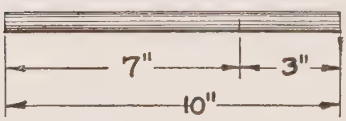


Fig. 16.

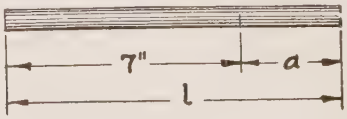


Fig. 17.

If the one part of the bar shown in Fig. 17 is 7 in. long and the other part is a in. long and the whole length l in.,

then $l = a + 7$ in.

and $l - 7$ in. $= a$

or $l - a = 7$ in.

Here the same process has been followed of transposing a term by changing its sign to the exact opposite, + to −.

Likewise, a negative term can be transposed by changing its sign from − to +.

If $a = l - 7$ in.
then $a + 7$ in. $= l$.

Hence, *any term may be transposed from one side of an equation to the other by changing its sign.*

This may also be explained in another way which is often quite useful. We can imagine that we have added or subtracted the same quantity from both sides of the equation. In the case of the formula $a = l - 7$ in. we can add 7 in. to both sides of the equation, or add 7 in. = 7 in. to the original equation

$$\begin{array}{rl} a & = l - 7 \text{ in.} \\ 7 \text{ in.} = & 7 \text{ in.} \\ \hline \text{adding,} \quad a + 7 \text{ in.} & = l \end{array}$$

This produces the same result as transposing the 7 in. Hence, we can say that: *If equal quantities are added to equal quantities, the sums are equal. If equal quantities are subtracted from equal quantities, the remainders are equal.*

In getting the circumference of a circle we multiply the diameter by π. If the circumference is given, we divide it by π to get the diameter.

$$C = \pi D$$
$$\text{or} \quad C \div \pi = D$$

Here we have in the first equation, π *times* a quantity on the right. We transposed the π by *dividing* it into the other side of the equation.

Likewise, if we have a quantity *divided* into the expression on one side, we can transpose it by *multiplying* it into the expression on the other side. We can thus transform $C \div \pi = D$ back into $C = \pi D$.

Now $\times$ is the opposite of $\div$, and $\div$ is the opposite of $\times$; hence, the same rule holds here, that we can transpose a quantity by changing its sign.

Here, also, we can state the principle in a different way. When we change $C = \pi D$ into $C \div \pi = D$, we can be said to have divided both sides of the equation by π. Likewise, when this equation is transformed back to $C = \pi D$, we can be said to have multiplied both sides by π.

Hence, *if equal quantities are multiplied by equal quantities, the products are equal. If equal quantities are divided by equal quantities, the quotients are equal.*

Example 1:

The law of levers states that "the force times the force arm equals the weight times the weight arm." From this derive a formula for the weight W with a weight arm b that can be lifted by a force P with a force arm a, as shown in Fig. 18.

$P\,a = W\,b$
$\div b \quad \div b$
$\frac{P\,a}{b} = \frac{W\,b}{b}$
$\frac{P\,a}{b} = W$
$W = \frac{P\,a}{b}$, *Answer.*

Explanation: The law can be written into the formula $P\,a = W\,b$. In order to get a formula for W we must transpose the b by which it is multiplied. To do this we divide both sides of the equation by b. This gives us $W \times \frac{b}{b}$ on the right. Now any quantity divided by itself is 1; hence, $\frac{b}{b} = 1$ and $W \times \frac{b}{b} = W \times 1 = W$. After getting a formula with W alone on the right we can rewrite the formula so as to exchange sides, without affecting the equality.

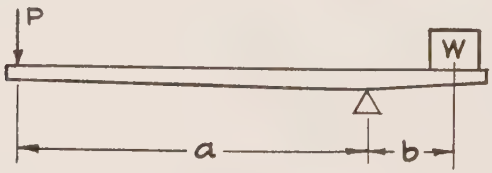


Fig. 18.

Example 2:

A keg contains a lot of rough blank nuts that are $1\frac{1}{4}$ in. square. What size bolts are they for?

$W = 1\frac{1}{2}\,D + \frac{1}{8}$
$W - \frac{1}{8} = 1\frac{1}{2}\,D$
$W - \frac{1}{8} = \frac{3}{2}\,D$
$\frac{2}{3} = \frac{2}{3}$
$\frac{2}{3}(W - \frac{1}{8}) = \frac{2}{3} \times \frac{3}{2}\,D$
$D = \frac{2}{3}\,(W - \frac{1}{8})$
Then $D = \frac{2}{3}(1\frac{1}{4} - \frac{1}{8})$
$= \frac{2}{3} \times 1\frac{1}{8}$
$= \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$ in., *Answer.*

Explanation: We have the formula for the width of a rough nut, $W = 1\frac{1}{2}\,D + \frac{1}{8}$ in., but must transform it into one having D alone on the left. First we transpose the $\frac{1}{8}$, leaving $1\frac{1}{2}\,D$ on the right. To get rid of the $1\frac{1}{2}$ or $\frac{3}{2}$ we can divide both sides by $\frac{3}{2}$ or multiply by $\frac{2}{3}$. Doing the latter and cancelling, we get D standing alone on the right. Next we exchange sides and have a formula for D and in this we substitute the width of the nuts and find that D is $\frac{3}{4}$ in. Hence, the nuts are for $\frac{3}{4}$-in. bolts.

There are two other operations sometimes performed on equations in order to transform them. It may be necessary to take a root of both sides or to raise both sides to some power.

If we take the formula for the area of a circle and transform it into a formula for D, we start with

$$A = .7854\,D^2$$

Divide both sides by .7854 and we have $\frac{A}{.7854} = D^2$ or, changing sides, $D^2 = \frac{A}{.7854}$.

Taking the square root of both sides, we get

$$\sqrt{D^2} = \sqrt{\frac{A}{.7854}}$$

But the square root of D^2 is D because $D \times D = D^2$.

Hence,
$$D = \sqrt{\frac{A}{.7854}}$$

As an example of the necessity of raising the terms of an equation to some power in order to effect a desired transformation, let us take the formula for the hypotenuse of a right triangle. According to the rule, the hypotenuse c of the right triangle in Fig. 19 would be

$$c = \sqrt{a^2 + b^2}$$

Now, suppose we know c and b and want to get the value of a. It would be necessary to transform this formula into a formula for a. As the formula now stands, we cannot transpose either a^2 or

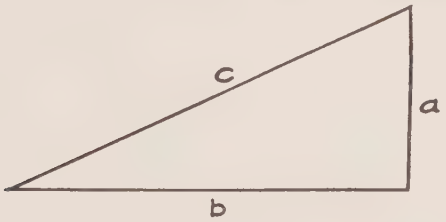


FIG. 19.

b^2 because they are covered by the square root sign. We can, however, get rid of this sign by squaring the equation. The square of c is c^2, and the square of $\sqrt{a^2+b^2}$ is a^2+b^2; that is, squaring the square root merely removes the square root sign, because the square root of a number is another number which *when squared* gives the original number. We now have

$$c^2 = a^2 + b^2$$

We can now transpose b^2, leaving

$$c^2 - b^2 = a^2$$

or
$$a^2 = c^2 - b^2$$

To get a formula for a we must take the square root of both sides of this equation, giving us

$$a = \sqrt{c^2 - b^2}$$

21. Rules of Transformation.—1. *A term may be transposed from one side of an equation to the other by changing its sign.*

2. *If equal quantities are added to equal quantities, the sums are equal.*

3. *If equal quantities are subtracted from equal quantities, the remainders are equal.*

4. *If equal quantities are multiplied by equal quantities, the products are equal.*

5. *If equal quantities are divided by equal quantities, the quotients are equal.*

6. *Like powers of equal quantities are equal.*

7. *Like roots of equal quantities are equal.*

22. Cancellation.—If the same term appears with the same sign in both sides of an equation, it may be cancelled or dropped from both without destroying the equality.

In the equation $y + 2\ a = 3\ x + 2\ a$ we could subtract $2\ a$ from both sides of the equation, leaving $y = 3\ x$. This is the same as cancelling or dropping the term $2\ a$ from both sides.

If both sides of an equation are multiplied or divided by the same quantity, that quantity can be cancelled from both sides without disturbing the equality.

For example, suppose we want to get the diameter D of a single pipe to have the same capacity as two smaller pipes d_1 and d_2. The area of the larger pipe must equal the sum of the areas of the smaller pipes, or

$$.7854\ D^2 = .7854\ d_1^2 + .7854\ d_2^2.$$

This can be written in the form

$$.7854\ D^2 = .7854(d_1^2 + d_2^2)$$

Here we have both sides of the equation multiplied by .7854. We can divide both sides by .7854 or, in other words, we can cancel .7854 from both sides, leaving

$$D^2 = d_1^2 + d_2^2$$

from which

$$D = \sqrt{d_1^2 + d_2^2}$$

The cancelling of the .7854 will save considerable useless multiplying and dividing by this number in using this formula.

It will be noticed that this formula is the same as that giving the relation between the sides of a right triangle if D is the hypotenuse and d_1 and d_2 are the other two sides. Hence, if we have two pipes and want to get the size of a single pipe of a capacity equal to the two, we can draw a right triangle with sides d_1 and d_2 as shown in Fig. 20. The hypotenuse D will give the size of the single pipe.

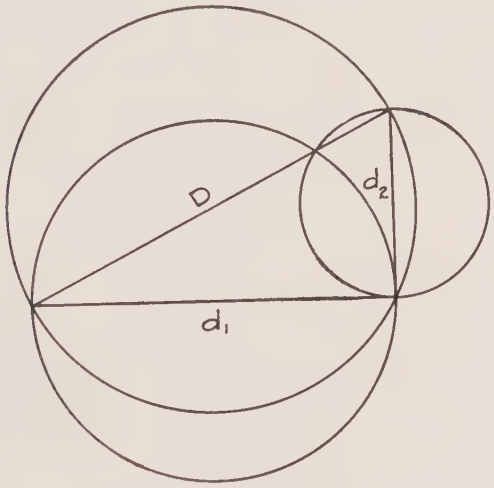


FIG. 20.

23. Changing Signs.—Sometimes, in transforming an equation, we get a formula for the desired quantity except that its sign is negative. In such a case we naturally want to change the sign of this term from $-$ to $+$. This can be done by changing the signs of all the terms in both sides of the equation. Thus, $-x=-2\,a+3$ can be changed to $x=2\,a-3$. This can be shown by changing all the terms to opposite sides of the equation, thus changing the signs.

$$-x=-2\,a+3$$
$$2\,a-3=x$$

Now we can simply change sides and get

$$x=2\,a-3$$

PROBLEMS

38. The cutting speed of the work in a lathe is equal to the circumference of the work times the number of revolutions per minute; or $S=CN$. Transform this to give an equation for N, the revolutions per minute. Then find

the R. P. M. necessary to give a cutting speed of 60 ft. per minute when the work is $3\frac{1}{2}$ in. in diameter.

39. The area of an ellipse is $A = .7854\ ab$, where a and b are the long and short diameters as shown in Fig. 21. Solve the formula for a and then find what the long diameter must be for an elliptical pipe to have an area of 33 sq. in. when the short diameter is 4 in.

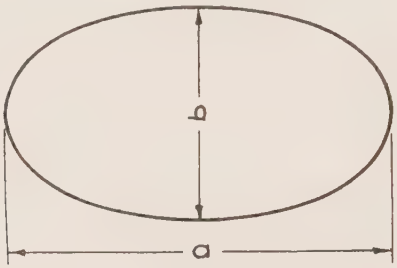


FIG. 21.

40. The wall surface of a room is $A = 2\ H(L+B)$, where H is the height from floor to ceiling and L and B are the length and breadth of a room. Solve for H and then find the height of a room 15×20 having 735 sq. ft. of wall surface.

41. Solve the formula for the area of a trapezoid, as given in Art. 6, Chapter I, to get a formula for the lower base b. Then find what the lower base must be to give an area of 85 sq. ft. when the upper base is 8 ft. and the height is 9 ft.

42. The horse-power of a belt is given as $H = \frac{PWV}{33000}$ where H represents the horse-power, P is the pull allowed per inch of width of the belt, W is the width of the belt in inches, and V is the velocity with which the belt is travelling in feet per minute. Transform this equation to get a formula to give the width of a belt needed for a given horse-power, at a given velocity, with a given allowable pull per inch of width. Then find the width of belt to transmit 50 horse-power at a velocity of 4500 ft. per minute with a pull of 70 lb. per inch of width.

43. A lot of blank hexagon U. S. standard nuts are $1\frac{3}{16}$ in. wide. What size bolts are they intended for?

44. The volume of a sphere or ball is $V = \frac{1}{6}\pi\ D^3$; transpose to get an equation for D.

45. Cast iron weighs .26 lb. per cu. in. Write a formula for the weight of a round cast iron ball of diameter D.

46. Transpose the formula obtained in problem 45 so as to have a formula for the diameter D of a cast iron ball of a given weight. Then find what the diameter would be for a ball to weigh 16 lb.

47. Write an equation showing the relation between the dimensions a and b of an elliptical pipe as shown in Fig. 21 and the diameter d of a round pipe having the same sectional area. Simplify the formula as far as possible

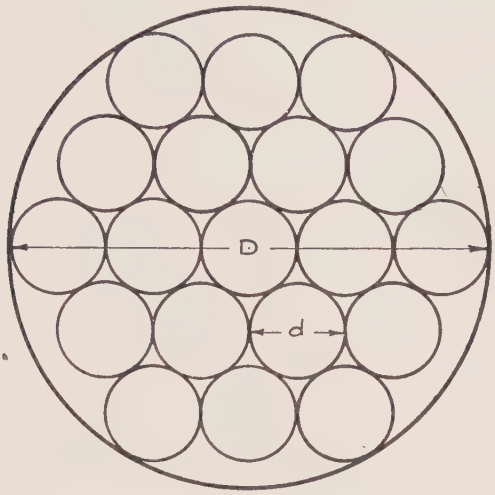


FIG. 22.

48. When a number of small circles are inscribed within a large circle as in Fig. 22, the number of the smaller circles can be obtained by the following approximate formula.

$$N = 0.907\left(\frac{D}{d} - 0.94\right)^2 + 3.7$$

Transform this into a formula giving the diameter D of the large circle to contain N small circles of diameter d. This is of use in calculating the size of a cable from the number and size of the wires that are to compose it.

49. A formula often used for helical springs made of round steel wire is

$$F = \frac{N\ W\ (D-d)^3}{1{,}500{,}000\ d^4}$$

where F, as shown in Fig. 23, is the amount that the spring will change in length when under a load W; D being the outside diameter of the spring, d the diameter of the wire of which the spring is made, and N being the number of turns or coils in the spring.

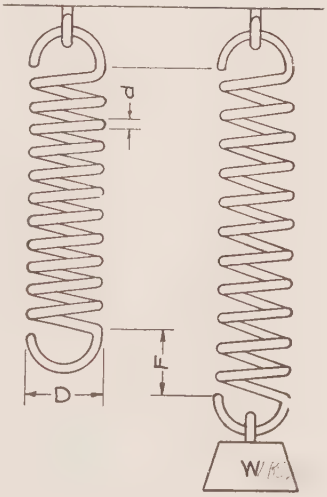


FIG. 23

Solve this formula for W and then write the resulting formula as a *rule*.

CHAPTER V

ALGEBRAIC MULTIPLICATION AND DIVISION

24.—Multiplication.—We multiply algebraic terms in practically the same way that we multiply denominate numbers in arithmetic:

$$\begin{array}{r} 5\ ft.\ 2\ in. \\ 3 \\ \hline 15\ ft.\ 6\ in. \end{array} \qquad \begin{array}{r} 12\ lb.\ 3\ oz. \\ 4 \\ \hline 48\ lb.\ 12\ oz. \end{array}$$

If letters are used in place of *ft.*, *in.*, *lb.*, or *oz.*, the multiplication will not be changed in the least:

$$\begin{array}{r} 5\ x+2\ y \\ 3 \\ \hline 15\ x+6\ y \end{array} \qquad \begin{array}{r} 12\ m+3\ n \\ 4 \\ \hline 48\ m+12\ n \end{array}$$

In arithmetic, the product of 3 ft. $\times$ 4 ft. is 12 sq. ft., or 12 ft.^{2}; replacing the *ft.* by a, it will be seen that $3\ a \times 4\ a = 12a^2$. Likewise, $a^4 \times a^2 = a \times a \times a \times a \times a \times a = a^6$;

$$3\ a^3 \times 2\ a^2b = 3 \times 2 \times a \times a \times a \times a \times a \times b = 6\ a^5b;$$

$$\pi\ D \times \frac{\pi D^2}{4} = \pi \times \pi \times D \times D \times D \times \frac{1}{4} = \frac{\pi^2 D^3}{4}.$$

An examination of these results shows that the methods of multiplying terms together can be summed up in the following statements or rules:

The coefficient of the product is the product of the coefficients of the terms to be multiplied together. The product will contain all the letters contained in the terms to be multiplied together. If the same letter appears in both terms, it will appear in the product with an exponent equal to the sum of its exponents in the two terms.

25. Division.—Since $a^2 \times a^4 = a^6$, it follows that $a^6 \div a^2 = a^4$, or that $a^6 \div a^4 = a^2$. Likewise, $6\ R^2 \div 2\ R = 3\ R$; $8\ a^2b \div 2\ ab = 4\ a$; $12\ x^2y^3 \div 4\ xy^2 = 3\ xy$.

When the dividend contains the same letters as the divisor, as in these cases, the division is very simple. When the terms are different, however, we can merely indicate the division. In arithmetic we learned that division could be expressed in the

form of a fraction; that $\frac{2}{7}$ meant the same as $2 \div 7$. Likewise, we can write division of algebraic terms as a fraction. $a \div b$ can be written $\frac{a}{b}$. In arithmetic we write the division as a fraction and then cancel or divide out any common factors that the terms may have, thus reducing the fraction to its lowest terms:

$$24 \div 32 = \frac{24}{32} = \frac{3 \times \cancel{8}}{4 \times \cancel{8}} = \frac{3}{4}$$

Likewise, in dividing algebraic terms, we can express the division as a fraction and in like manner can cancel any common factors:

$$18\ a^3b^2c \div 6\ a^2c = \frac{\overset{3a}{\cancel{18}\cancel{a}^3\ b^2\cancel{c}}}{\cancel{6}\cancel{a}^2\cancel{c}} = 3\ ab^2$$

$$4\ x^2y \div 2\ x^3y = \frac{\overset{2}{\cancel{4}\cancel{x}^2\cancel{y}}}{\underset{x}{\cancel{2}\cancel{x}^3\cancel{y}}} = \frac{2}{x}$$

26. Law of Signs.—In either multiplying or dividing, *like signs give* $+$ and *unlike signs give* $-$.

That the product or quotient of two positive quantities is positive needs no demonstration, but there may be some doubt as to just how to handle negative terms. One or two simple illustrations will show these relations.

If the thermometer reads 6° below zero and the mercury then goes down to twice as far, it will stand at 12° below:

$$-6\ d \times 2 = -12\ d$$

Hence, a negative number times a positive gives a negative product.

If the thermometer is 6° below zero and then suddenly changes to twice as much *in the other direction* from zero, it will read 12° above zero.

$$-6\ d \times (-2) = +12\ d$$

Hence the product of two negative quantities is positive.

By transposing these two equations we can prove any other cases of multiplication or division:

Multiplication	Division
$(+6\ d) \times (+2) = +12\ d$	$(+12\ d) \div (+2) = +6\ d$
$(-6\ d) \times (+2) = -12\ d$	$(-12\ d) \div (+2) = -6\ d$
$(+6\ d) \times (-2) = -12\ d$	$(-12\ d) \div (-2) = +6\ d$
$(-6\ d) \times (-2) = +12\ d$	$(+12\ d) \div (-2) = -6\ d$

27. Multiplication of Polynomials.—In multiplying an expression containing two or more terms by a single term, we naturally multiply each term of the expression by the multiplier:

Examples:

1.
$$\begin{array}{r} 4\,a^2+4\,ab+b^2 \\ \hline a_2 \\ \hline 4\,a^4+4\,a^3b+a^2b^2 \end{array}$$

2.
$$\begin{array}{r} h_1+h_2 \\ \hline \pi\,r^2 \\ \hline \pi\,r^2h_1+\pi\,r^2h_2 \end{array}$$

3.
$$\begin{array}{r} D^2-d^2 \\ \hline \pi \\ \hline \pi\,D^2-\pi\,d^2 \end{array}$$

4.
$$\begin{array}{r} x^2-2\,x+1 \\ \hline -x \\ \hline -x^3+2\,x^2-x \end{array}$$

It is sometimes necessary to multiply one algebraic expression by another expression consisting of two or more terms. In such a case the first expression is multiplied separately by each term of the second, and the resulting terms are added. The method is best shown by a few examples which should be studied carefully. It will be noticed that the work is arranged very much as in arithmetic. Like terms in the products are arranged in vertical columns so they can be added to get the total product.

Example 1:

$$(a^2+2\,a+1)\,(a-1)=?$$

$$\begin{array}{r} a^2+2\,a+1 \\ \hline a-1 \\ \hline a^3+2\,a^2+\ a \\ -\ a^2-2\,a-1 \\ \hline a^3+\ a^2-\ a-1 \end{array}$$

Explanation: Having written down the multiplicand and the multiplier, we multiply the multiplicand by the first term, a of the multiplier. This gives $a^3+2\,a^2+a$. We then multiply by the next term, -1. Since the sign of this term is $-$, the products will all have their signs different from those in the multiplicand. This gives us $-a^2-2\,a-1$, which we write so that the a^2 term will be under the a^2 term in the first product and the a term under the a term. We now find the algebraic sum of all the terms in these separate products, and get a^3+a^2-a-1 as our final product.

Example 2:

$$(a-b)\,(c-d)=?$$

$$\begin{array}{r} a-b \\ \hline c-d \\ \hline ac-bc \\ -ad+bd \\ \hline ac-bc-ad+bd \end{array}$$

Explanation: There is one particular point to be noticed here. There are no like terms in the two partial products; hence, the second partial product is placed at one side and the final or total product is merely an expression of the algebraic addition between all the terms of the two partial products.

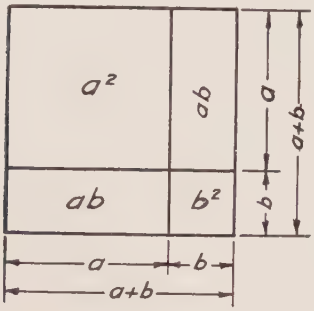
3. $(x^2+y^2)\,(x^2-y^2)=?$

$$\begin{array}{r} x^2+y^2 \\ \hline x^2-y^2 \\ \hline x^4+x^2y^2 \\ -x^2y^2-y^4 \\ \hline x^4\qquad -y^4 \end{array}$$

4. $(2\,x^3-3\,x^2+4\,x-5)\,(x^2-2\,x-3)=?$

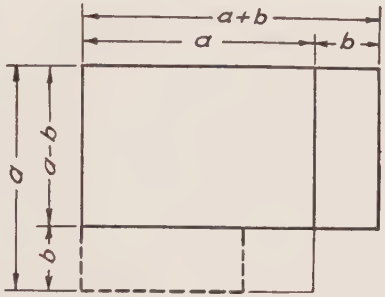
$$\begin{array}{r} 2x^3-3\,x^2+4\,x-5 \\ \hline x^2-2\,x-3 \\ \hline 2x^5-3\,x^4+4\,x^3-5\,x^2 \\ -4\,x^4+6\,x^3-8\,x^2+10\,x \\ -6\,x^3+9\,x^2-12\,x+15 \\ \hline 2x^5-7x^4+4\,x^3-4\,x^2-2\,x+15 \end{array}$$

Many of these multiplications of binomials can be shown or pictured by figures if we assume the letters to represent lengths of lines. For example $(a+b)^2$ can be represented by a square, Fig. 24, measuring $a+b$ on each side. The product of $(a+b) \times (a+b)$ is $a^2+2\ ab+b^2$. The figure shows this quite plainly, the area being made up of one square $a \times a$ or a^2 in area, another square b^2 in area, and two rectangles each ab in area; or, altogether, the area is $a^2+2\ ab+b^2$.



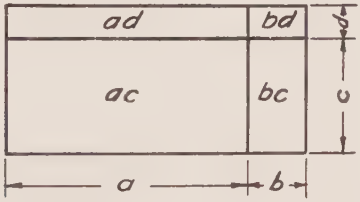
$$(a+b)^2 = a^2 + 2ab + b^2$$

Fig. 24.



$$(a+b)(a-b) = a^2 - b^2$$

Fig. 25.



$$(a+b)(c+d) = ab + bc + ad + bd$$

Fig. 26.

Fig. 25 illustrates how $(a+b)(a-b)$ can be shown to be a^2-b^2. The area in heavy lines is $a+b$ long and $a-b$ high; so that its area represents the product of these two expressions. If the small rectangle at the right is removed and placed below the larger one, as shown by the dotted lines, it will be seen that the area lacks a small square, b^2, of being equal to a^2. In other words the product is a^2-b^2.

Fig. 26 shows the product of $(a+b)(c+d)$ which is $ac+bc+ad+bd$.

Fig. 27 illustrates a volume having the dimensions $(a+2\ c)$, $(a+c)$, and b. As shown, the block consists of 1 piece having a volume a^2b, 3 pieces each abc, and 2 pieces each bc^2. This gives the total volume $a^2b+3\ abc+2\ bc^2$, which checks with the result obtained by multiplying the three expressions represented by the dimensions of the block.

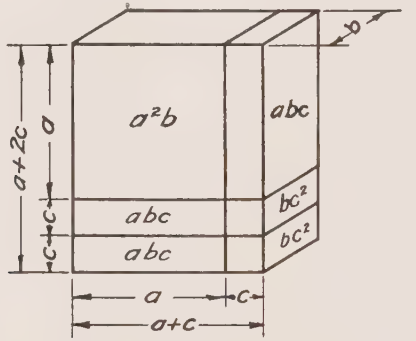


Fig. 27.

28. Division of Polynomials.—When an expression of two or more terms is to be divided by a single term, the work can be arranged as in short division in arithmetic.

Examples:

$$2)\overline{10\text{ lb. }6\text{ oz.}}$$
$$5\text{ lb. }3\text{ oz.}$$

$$2)\overline{10\ x+6y}$$
$$5\ x+3y$$

$$2\ x)\overline{2\ x^3-6x^2-10\ x}$$
$$x^2-3x\ -\ 5$$

$$-a)\overline{\ a^2-2\ ab+b^2}$$
$$-a\ +2\ b\ -\frac{b^2}{a}$$

When an expression of two or more terms is to be divided by another expression of two or more terms, the work is arranged and carried on as in long division in arithmetic.

Examples:

1. Divide $x^2+2\ x+1$ by $x+1$

$$x+1)\overline{x^2+2\ x+1}(x+1$$
$$\underline{x^2+\ \ x}$$
$$\quad\ x+1$$
$$\quad\ \underline{x+1}$$

Explanation: Dividing the first term (x) of the divisor into the first term (x^2) of the dividend, we get (x) as the first term of the quotient. We write this as the first term of the quotient and multiply it into the divisor just as in arithmetic:

$$x\times(x+1) \text{ or } x(x+1)=x^2+x;$$

we place this product under the dividend and subtract, as in arithmetic, and bring down the next term.

Next, we divide the first term of the divisor (x) into the first term of this remainder (x) and get 1; we place the 1 in the quotient, and then multiply it into the divisor:

$$1\times(x+1) \text{ or } 1(x+1)=x+1$$

We place this product under the remainder, $(x+1)$, and subtract. There is no remainder and, since there are no more terms in the dividend, the division is complete.

2. $(x^2-8\,x+16)\div(x-4)=?$

$$\begin{array}{l} x-4)x^2-8\,x+16\,(x-4 \\ \quad\underline{x^2-4\,x} \\ \quad\quad -4\,x+16 \\ \quad\quad \underline{-4\,x+16} \end{array}$$

3. $\dfrac{(x^3+3\,x^2+3\,x+1)}{x+1}=?$

$$\begin{array}{l} x+1)\,x^3+3\,x^2+3\,x+1\,(x^2+2\,x+1 \\ \quad\underline{x^3+\;\;x^2} \\ \quad\quad 2\,x^2+3\,x \\ \quad\quad \underline{2\,x^2+2\,x} \\ \quad\quad\quad\quad x+1 \\ \quad\quad\quad\quad \underline{x+1} \end{array}$$

4. $\dfrac{a^2-2\,ab+b^2}{a-b}=?$

$$\begin{array}{l} a-b)\,a^2-2\,ab+b^2\,(a-b \\ \quad\underline{a^2-\;\;ab} \\ \quad\quad -\;\;ab+b^2 \\ \quad\quad \underline{-\;\;ab+b^2} \end{array}$$

5. $\dfrac{a^3+3\,a^2b+3\,ab^2+b^3}{a+b}?$

$$\begin{array}{l} a+b)\,a^3+3\,a^2b+3\,ab^2+b^3\,(a^2+2\,ab+b^2 \\ \quad\underline{a^3+\;\;a^2b} \\ \quad\quad\quad 2\,a^2b+3\,ab^2 \\ \quad\quad\quad \underline{2\,a^2b+2\,ab^2} \\ \quad\quad\quad\quad\quad ab^2+b^3 \\ \quad\quad\quad\quad\quad \underline{ab^2+b^3} \end{array}$$

Unless a division of one expression by another comes out even, that is, without a remainder, it is much better to leave the division expressed as a fraction rather than to attempt to perform the division.

29. Factoring.—Factoring is quite useful in simplifying formulas and will usually save considerable labor in working with them. An expression for the sum of the circumferences of two circles can be written $\pi D+\pi\,d$,but since both diameters are multiplied by the same factor, π, we can write the expression $\pi(D+d)$. In substituting certain values of D and d in this expression as it now stands, there is only one operation of multiplication to be performed, instead of two as in the original form.

When two or more terms of an expression contain a common letter or number, we can remove this factor, if desired, and place it outside a parenthesis containing what remains of the terms:

$$\pi\,D+\pi\,d=\pi(D+d)$$
$$.7854\,D^2-.7854\,d^2=.7854(D^2-d^2)$$

In these simple cases the factor removed and the terms left can be seen at a glance. In more difficult cases the common factor is

found by inspection and the remaining terms are found by dividing the original expression by the factor to be removed.

Example 1:

Factor the expression $3\ m^3+3\ m^2n+3\ mn^2$

The common factor of all the terms is $3\ m$

$$3\ m)\ 3\ m^3+3\ m^2n+3\ mn^2$$
$$m^2+\ mn\ +\ \ n^2$$

Hence $3\ m^3+3\ m^2n+3\ mn^2=3\ m\ (m^2+mn+n^2)$

Example 2:

Factor $ac+bc+ad+bd$

There is no common factor of all the terms but we can factor c out of the first two terms and d out of the last two. This gives:

$$c(a+b)+d(a+b)$$

Now we can factor $(a+b)$ out of the expression, giving us $(a+b)(c+d)$.

PROBLEMS

Multiply:

50. 6 ft. 5 in. by 2;
$6x+5\ y$ by 3.

51. $2\ a+3\ b$ by $2\ b$.

52. $5m-3\ n$ by $-2\ p$.

53. $a+2\ b$ by $a+b$.

54. $a-3\ b$ by $a-b$.

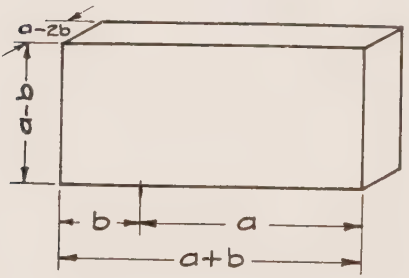


Fig. 28.

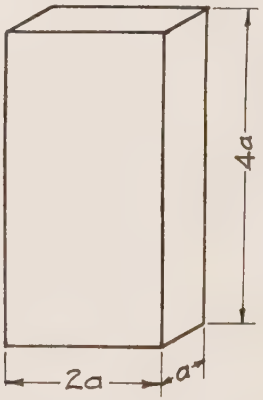


Fig. 29.

Divide:

55. 27 ft. by 3 ft.

56. 27 a by 3 a.

57. 16 *sq. ft.* by 2 *ft.*;
16 a^2 by 2 a.

58. $3\ a^2b^3$ by ab.

59. $3\ a^2b-2\ ab^2$ by ab.

60. $3\ x-4\ xy+xz$ by $-x$.

61. a^2-a-12 by $a+3$.

62. $x^3-6\ x^2+12\ x-8$ by $x-2$.

Factor:

63. $\frac{\pi}{4} D^2 - \frac{\pi}{4} d^2$.

64. $.283\ a^3 + .283\ a^2 b$.

65. $\pi\ r^2 a + \pi\ r^2 b$.

66. $\frac{\pi}{4} D^2 l - \frac{\pi}{4} d^2 l$.

67. A rectangular block, as shown in Fig. 28, has the following dimensions: length, $a+b$, breadth, $a-b$, thickness, $a-2\ b$. What is the volume of the block?

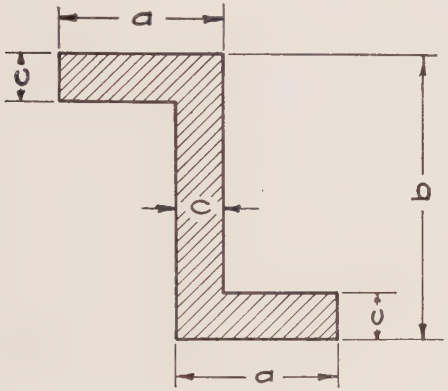


Fig. 30.

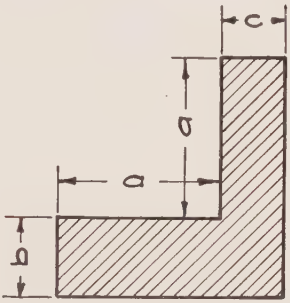


Fig. 31.

68. A billet of steel is a inches thick, 2 a inches wide, and 4 a inches high, as shown in Fig. 29. What are the dimensions if the weight is 2000 lb. (1 cu. in. of steel = .283 lb.)?

69. Work out a formula for the area of the Z-shaped figure in Fig. 30.

70. Work out a formula for the area of the L-shaped figure in Fig. 31.

CHAPTER VI

SOLUTION OF SIMPLE EQUATIONS

30. Formation of Equations.—One of the greatest uses of a knowledge of formulas or algebra is in solving problems that are rather complicated and could not be easily solved by arithmetic. Sometimes the statement of a problem and of the relation between the unknown and known quantities is so complex that it is difficult to figure out the operations necessary to solve the problem by arithmetic. In such a case, the statement of the problem can be written as an equation or formula, using a letter to represent the unknown quantity. We can then transpose the equation to find the value of this letter.

Example:

A man works a full day of 10 hr. and in addition works $3\frac{1}{2}$ hr. overtime, for which he receives "time and a half." If he is paid \$3.66 for the entire time, what is his rate per hour?

Let r = his rate per hour.

Then, $10\,r + 3\frac{1}{2} \times 1\frac{1}{2}\,r = \$3.66$, or $366¢$

$10\,r + \frac{7}{2} \times \frac{3}{2}\,r = 366¢$

$10\,r + 5\frac{1}{4}\,r = 366¢$

$15\frac{1}{4}\,r = 366¢$

$r = 366 \div 15\frac{1}{4} = 24¢$, *Answer.*

Explanation: By using the letter r to represent his rate, we can calculate his pay for the day in terms of r. For the regular day of 10 hr. he receives $10 \times r$ or $10\,r$; for each hour overtime he receives $1\frac{1}{2}\,r$; or for the $3\frac{1}{2}$ hr. he receives $3\frac{1}{2} \times 1\frac{1}{2}\,r$, or $5\frac{1}{4}\,r$. His whole pay is therefore $15\frac{1}{4}\,r$, and this, the problem states, is equal to 366 cents. By dividing both sides of this equation by $15\frac{1}{4}$ we find that $r = 24¢$.

By this process such problems can be worked much more easily than by arithmetic.

In working problems by algebra, the first step is to assign some letter to represent the unknown item in the problem. Unless there is a special reason for using some particular letter, it is customary to use the letter x.

The next and most important step is the *formation* of the equation; that is, the construction of an equation or formula from the written statement of the problem. In a great many cases the statement of the problem is such that the equation can be formed directly from it by replacing words by their corresponding

mathematical signs, just as we write a formula from a given rule. In any case we must get a formula or equation that expresses the relation given in the problem between the unknown and the known items.

One of the greatest difficulties experienced by the beginner in the use of algebra is in changing the statement of the problem into an equation. The following illustrations will aid greatly in getting an idea of the method to be followed.

If x represents a certain unknown quantity, then 3 times that quantity would be represented by $3\,x$; one-half of the quantity would be represented by $\frac{x}{2}$; while the quantity squared would be represented by x^2.

If one man's pay is represented by x, then that of a man who gets $1\frac{1}{2}$ as much would be $1\frac{1}{2}\,x$. If a train travelled for a hours at a rate of x miles per hour, the total distance travelled would be ax miles.

The following examples show how the equations are formed from the statements of problems.

Example 1:

A man sells half his interest in a factory, and later sells $\frac{1}{4}$ of what he has left. His interest is then worth \$75,000. How much was his original interest worth?

Let x = his original interest.

In the first sale he sells $\frac{1}{2}\,x$ and has left $\frac{1}{2}\,x$. Then he sells $\frac{1}{4}$ of $\frac{1}{2}\,x$, which is $\frac{1}{8}\,x$. This leaves him $\frac{1}{2}\,x-\frac{1}{8}\,x$, or $\frac{3}{8}\,x$, which is worth \$75,000. Then

$$\tfrac{3}{8}\,x = \$75{,}000$$

Or we could write from the beginning

$$x-\tfrac{1}{2}\,x\ -\tfrac{1}{4}\,(x-\tfrac{1}{2}\,x) = \$75{,}000$$

which also gives

$$\tfrac{3}{8}\,x = \$75{,}000.$$

From either of these equations we can find the value of x, his original holdings. Dividing both sides of the equation by $\frac{3}{8}$, we get $x = \$200{,}000$.

Example 2:

Two boys buy a launch for \$250. If one of them pays \$25 more than the other and they agree to use the launch in proportion to what they pay, how many days should each use it out of every 20 days?

Let x = what one boy pays.

Then, $x+\$25$ = what the other one pays

and, $x+x+\$25 = \$250$

$2\,x+\$25 = \$250$

Transposing the \$25, we get

$$2\,x = \$250-\$25 = \$225.$$

Now, dividing both sides of the equation by 2, we have

$$x = \$112.50.$$

This is what one paid; while the other one paid $x+\$25$ or $\$112.50+\$25 = \$137.50$.

Then one should use the boat $\frac{112\frac{1}{2}}{250}$ or $\frac{225}{500}$ or $\frac{9}{20}$ of the time, and the other one $\frac{137\frac{1}{2}}{250}$ or $\frac{275}{500}$ or $\frac{11}{20}$ of the time. Then one should use it 9 days, and the other 11 days out of the 20 days.

31. Solution of Equations.—Having formed an equation from the statement of a problem, the next step is so to transform it as to get the value of the unknown item. This involves using the principles of transformation of formulas given in Chapter IV and is usually called *solving the equation.*

Suppose we have the equation $5\,x-10=3\,x+6$. Solving the equation would mean finding the value of x. To do this we transpose all the terms containing x to the *left* side of the equation and all other terms to the *right.* In this case we get

$$5\,x-10=3\,x+6$$
$$5\,x-3\,x=10+6$$

Collecting and adding the like terms, this becomes

$$2\,x=16$$

and dividing both sides of the equation by 2 we get

$$x=8.$$

The value of x is therefore 8. The correctness of the work can be tested by substituting this value for x in the original equation. If, with this value of x, the two sides are equal, then the solution is correct. Substituting 8 for x in the original equation

$$5\,x-10=3\,x+6$$
$$\text{we get } 5\times 8-10=3\times 8+6$$
$$40-10=24+6$$
$$30=30$$

Since the two sides give the same number, the value of x must be correct.

From this example it will be seen that, after the equation for a problem has been formed, the procedure for finding the unknown quantity can be summed up in three brief rules.

1. *Transpose all terms containing x to one side of the equation and all other terms to the other side.*
2. *Combine the terms in each side of the equation.*
3. *Divide both sides by the coefficient of x.*

Every formula is an equation and likewise every equation can be considered as a formula and its terms transposed as explained in Chapter IV. If an equation or formula contains two different letters, we cannot get the numerical values of *both* from a *single equation*, unless we have some additional information regarding the relation between them. We can, however, solve the equation for one of the values in terms of the others. This is what was done in Chapter IV when we transformed an equation so as to get a formula for a certain letter in terms of the others.

If a problem contains one unknown quantity, its value can be found from a single relation or equation.

If a problem contains two unknown quantities, their values can be found if there are two separate relations given, so that two equations can be formed or so that one unknown can be expressed in terms of the other. For example, in the problem about the launch in Art. 30 there were two unknown quantities, the amounts that each paid, but one amount could be expressed in terms of the other so that we only had one letter in the equation for the problem. There were really two relations given for the two amounts paid; first, that one was \$25 more than the other; and, second, that the sum of the two amounts was \$250.

If there are three unknowns, there must be three relations given; and so on, always as many relations being necessary as there are unknown quantities to be found.

Having reduced the stated conditions of a problem to a formula or equation, the solution of the problem then depends on the principles of transformation given in Art. 21, Chapter IV.

In solving problems by algebra, study over carefully the statement of the problem until you are able to write out an equation representing the relations, letting x or some other letter stand for the unknown quantity. In writing your solution, always state first what x is to stand for. Next write out the equation and then solve the equation for the unknown quantity. Before accepting this value as correct, return to the original statement of the problem and see if this value fulfills the given conditions.

PROBLEMS

71. Find the value of x in each of the following equations:

(a) $2x-7=5$ *Answer*, $x=6$

(b) $x+3=1$ *Answer*, $x=-2$

(c) $2x-4=x-1$ *Answer*, $x=3$

(d) $5x-7=6x-9$ *Answer*, $x=2$

72. Find the value of the unknown quantity in each of the following equations:

(a) $.7854\ D^2 = 113.1$ *Answer*, $D = 12$

(b) $\frac{\pi}{6} D^3 = 250$

(c) $\frac{6(a+8)}{2} = 60$

(d) $\sqrt{c^2 - 12^2} = 7$

73. A moulder working on piece work loses 8 per cent. of his castings from defects. If he is paid 6 cents each for the good castings and gets $8.28, how much money does he lose due to the defective castings?

74. The sales of a firm increased 15 per cent. the second year over the first, and the third year they were 25 per cent. more than they were the second year. If their sales totaled $359,375.00 the third year, how much were they the first year?

75. A manufacturer makes 16 per cent. profit over the cost of manufacture of a certain article. He sells to a jobber who adds a profit of 10 per cent. to what he pays the manufacturer. The retailer then adds 40 per cent. to what he pays the jobber and sells the article for $5.75. What was the original manufacturing cost?

76. The fastest speed of the spindle of a lathe with the back gears in is 66 revolutions per minute. What is the diameter of a casting that would give a cutting speed of 40 ft. a minute when turned at this speed?

77. From a tank one-half full of crude oil, 1000 gal. are drawn and 75 gal. are lost by evaporation and leakage. The tank is then one-third full. How much does the tank hold when full?

78. A factory increases its force of men by 40 per cent. and increases its working hours per week from 48 to 60. If it is then able to turn out 210 machines a week, how many did it turn out before the change?

79. A machinist and his helper are pro-rated on a job, which means that the money they earn together is divided according to their day-work rates. If the machinist's rate is $2\frac{1}{2}$ times that of his helper and they together earn $42.80, how much does each get?

80. A lot of brass scrap weighing 600 lb. contains 30 per cent. zinc. How many pounds of zinc must be added in melting to increase the percentage of zinc to 34 per cent.? (Remember that the zinc added will also increase the total weight of metal.)

CHAPTER VII

SIMULTANEOUS EQUATIONS; QUADRATIC EQUATIONS

32. Simultaneous Equations.—If a single equation contains two unknown quantities, it is not possible to find the value of either of them except in terms of the other. The value of either quantity depends on the value of the other one. For example, in the equation $x+3y=17$, y might have any value and x would have certain corresponding values. Transposing the $3y$, we get $x=17-3y$.

If, $y=1$, then $x=17-3\times 1=14$
$y=2$, then $x=17-3\times 2=11$
$y=5$, then $x=17-3\times 5=2$

In the same way, if values are given to x, corresponding values can be found for y, but there is nothing about the single equation that says that either x or y must have certain definite values; it merely states a relation between them.

However, if we have a second equation giving another relation between x and y for the same problem, we can then determine definitely from the two equations the exact values of x and y. If, in the case just given, we also knew that $2x+y=9$, we could transform it into $2x=9-y$. Assigning to y the same values as before,

if, $y=1$, then $x=\frac{9-1}{2}=4$

$y=2$, then $x=\frac{9-2}{2}=3.5$

$y=5$, then $x=\frac{9-5}{2}=2$

If we compare the two sets of values for x and y, it will be seen that the only pair of values for x and y that will fit *both* equations is $x=2$ and $y=5$. These are therefore the *simultaneous* values of x and y for the two equations, and the two equations are called *simultaneous equations.*

To solve simultaneous equations, we must have as many distinct relations or equations as there are unknowns to be found.

If there are two unknown quantities, we must, as just shown, have two equations. If there are three unknown quantities, there must be three equations before we can get the values of the unknown numbers; and so on, always as many equations as there are unknown quantities. To take a practical illustration; it is easy to see that simply knowing the proper surface speed for emery wheels will not tell us both the size of a wheel and the proper R. P. M. (revolutions per minute) to run it. If we know the size of wheel to be used, then we can figure the proper R. P. M. for it; or if we know the R. P. M. of an arbor, we can calculate the size of wheel to order for it.

33. Solving by Substitution.—Some of the problems of the preceding chapter might be classed as cases where there were two unknown numbers; but, instead of using different letters for the numbers, we got one number in terms of the other. Then the equation with which we worked contained only one unknown letter and so only one equation was necessary. We might, however, have used two letters for the two unknown quantities and thus have obtained two equations. For example, take the problem in Article 30 about the two boys who bought the launch for \$250, one of them paying \$25 more than the other. If we let x = what one boy paid and y = what the other paid, we would get the two equations:

$$x+y=\$250$$
$$y=x+\$25$$

Here we have two equations stating the two relations between x and y; first, that the sum of the two amounts was \$250, and, second, that one was \$25 more than the other. Now, in place of y in the first equation we can substitute its value, $x+\$25$, as given in the second equation. This changes the first equation to $x+(x+\$25)=\$250$; and from this we can calculate x just as was done before. Then we substitute for x in either equation its value as just found and solve for the value of y.

This method is used principally where the equations are rather simple and where one equation will give one letter in terms of the other in very simple form.

To further illustrate the method, we will take the following simultaneous equations:

$$x+y=15$$
$$x-y=3$$

These equations would state in words that the sum of two quantities is 15 and their difference is 3. Take the second equation and transpose y to the right hand member. $x-y=3$ becomes $x=y+3$. If $x=y+3$, we can substitute $y+3$ for x in the first equation.

$$\begin{aligned} x+y&=15\\ y+3+y&=15\\ 2\,y&=15-3=12\\ y&=12\div 2=6 \end{aligned}$$

Taking this value of y $(=6)$ we can substitute it for y in either original equation and find the value of x

$$\begin{array}{lll} x+y=15 & \text{or} & x-y=3\\ x+6=15 & & x-6=3\\ x=9 & & x=9 \end{array}$$

Hence, for these two equations, x equals 9, and y equals 6.

34. Solving by Elimination.—Another method of solving simultaneous equations is to multiply or divide both equations so as to make the coefficients of one of the unknowns the same in the two equations. Then by adding or subtracting the two equations we can get an equation containing only one unknown quantity, the value of which we can readily find in the usual way. For example, take the two equations:

$$\begin{aligned} 2\,x+5\,y&=25 \qquad (1)\\ 3\,x-2\,y&=9 \qquad (2) \end{aligned}$$

If we multiply equation (1) by 3 and equation (2) by 2, we will make the coefficients of x the same in both the resulting equations.

$$\begin{aligned} 6\,x+15\,y&=75\\ 6\,x-\ 4\,y&=18 \end{aligned}$$

Subtracting,

$$\begin{aligned} 19\,y&=57\\ y&=57\div 19=3 \end{aligned}$$

Substituting this value of y in (2) we have

$$\begin{aligned} 3\,x-2\times 3&=9\\ 3\,x&=9+6=15\\ x&=15\div 3=5 \end{aligned}$$

Hence, for these two equations $x=5$ and $y=3$. This is proved by substituting these values in the original equations to see if the equality is maintained with these values for x and y.

$$\begin{aligned} 2\times 5+5\times 3&=25\\ 3\times 5-2\times 3&=9 \end{aligned}$$

The values maintain the equality and therefore are the correct values. The following example will show how the two equations are formed and solved for a practical problem.

Example:

If we assume that scrap iron contains 2 per cent. silicon and we have a pig iron containing 6 per cent. silicon, how many pounds of each should be used to make a ton (2000 lb.) of mixture containing 3.2 per cent silicon?

Let x = weight of scrap iron to be used,
and y = weight of pig iron to be used.
Then $.02\,x$ = pounds of silicon in the scrap used,
$.06\,x$ = pounds of silicon in the pig used.
$.032 \times 2000 = 64$ lb. of silicon needed in the mixture.

$$.02\,x + .06\,y = 64 \qquad (1)$$
$$x + y = 2000 \qquad (2)$$

If we multiply (1) by 50, we will make its coefficient of x the same as that in (2). We can then subtract one from the other and eliminate x.

Multiplying (1) by 50, $x + 3\,y = 3200$
Bringing down (2) $x + y = 2000$
Subtracting $2\,y = 1200$
$y = 600$ lb.

Having the value of y we can substitute it in (2), giving

$x + 600 = 2000$
$x = 2000 - 600 = 1400$ lb.

Hence, we should use 1400 lb. of scrap and 600 lb. of pig to get the desired mixture.

35. Equations of More Than Two Unknowns.—As previously stated, there must always be as many equations as there are unknowns if the values of the unknowns are to be found. The method of elimination is the usual method followed in solving such problems. Let us take these three equations for x, y, and z to show the method used.

$$x + y + 2\,z = 13 \qquad (1)$$
$$2\,x - y - z = 4 \qquad (2)$$
$$x - y + 2\,z = 5 \qquad (3)$$

We must plan to eliminate one of the letters by the method just shown. Let us plan on eliminating y. If we add equations (1) and (2), y will not appear in the resulting equation. Likewise adding (3) and (1) will eliminate y.

Adding (1) and (2), $3\,x + z = 17 \qquad (4)$
Adding (1) and (3), $2\,x + 4\,z = 18 \qquad (5)$

Now we have two equations containing two unknowns. Multiplying (4) by 4 and subtracting (5) from it will eliminate z.

Multiplying (4) by 4, $12x+4z=68$

(5) $2x+4z=18$

Subtracting, $10x=50$

$x=50\div 10=5$

Having the value of x, we can return to equation (4) and substitute this value for x and find z.

$$3\times 5+z=17$$
$$z=17-15=2$$

Now we have $x=5$ and $z=2$. Returning to one of our original equations, say (1), we can substitute these values and find y.

$$5+y+2\times 2=13$$
$$y=13-5-4=4.$$

Hence $x=5$, $y=4$, and $z=2$.

36. Quadratic Equations.—When an equation contains the *square* of an unknown quantity, it is called a *quadratic* equation. If the equation contains only the square of the unknown, the solution is simple. In the equation $x^2-8=1$, we can transpose the -8, giving

$$x^2=9$$

This is easily solved for the value of x by extracting the square root of both sides, giving

$$x=\pm 3$$

The double sign ($\pm$) means that this quantity may be either "*plus or minus.*" In Chapter V it was shown that the product of two minus quantities is a positive quantity. Hence, 9 is either the product of $+3\times+3$ or of -3×-3 and, therefore, the value of x in this equation is ± 3, or *plus or minus* 3. In very few practical problems would we have to consider the minus quantity, but it must be understood that every quadratic equation will give two possible values of the unknown quantity in the same manner. The practical considerations of the problem will indicate which value is the one sought.

When an equation has terms containing x^2 and also terms containing x, the finding of the square root is a little more complicated. If an expression like $x+3$ is squared, the result is x^2+6x+9; if $x-5$ is squared, the result is $x^2-10x+25$; likewise, with any similar expression, its square is made up of three terms, the square of the first term, twice the product of the first by the

second, and the square of the second term. Then working from the square of such a term we can find the square root.

$$\sqrt{x^2+14\,x+49} = x+7$$
$$\sqrt{x^2-12\,x+36} = x-6$$
$$\sqrt{y^2-2\,ay+a^2} = y-a$$

Now, suppose we have an equation like

$$x^2+6x+5=8$$

The expression on the left of the = sign is not the square of any binomial expression. Dividing the 6 by 2 will give the term of which the third term should be the square. This expression should, therefore, be $x^2+6\,x+9$ to be a perfect square of $x+3$. Consequently it is necessary to add something to both sides of the equation before we can get the square root. To make the work simpler we will transpose the 5 to the right, giving

$$x^2+6\,x=8-5=3$$

To make the expression $x^2+6\,x$ a perfect square of $x+3$ we must add 9, thus giving $x^2+6\,x+9$, which is the exact square of $x+3$. This 9 must be added to both sides of the equation to preserve the equality, giving

$$x^2+6\,x+9=9+3, \text{ or}$$
$$x^2+6\,x+9=12$$

Now, we can get the square root of both sides of the equation, giving

$$x+3=\pm\sqrt{12}=\pm 3.464$$

Next we can transpose the 3 to the other side of the equation and get the value of x.

$$x=-3\pm 3.464$$

The two possible values of x are therefore $-3+3.464$ and $-3-3.464$; or $+0.464$ and -6.464. The practical considerations of the problem would show which is the value wanted.

The method of solving quadratic equations can be summed up as follows:

(1) Transpose all terms containing x^2 and x to the left side of the equation and all other terms to the right hand side.

(2) Simplify as far as possible and divide the equation, if necessary, in order to make the coefficient of x^2 unity.

(3) Add to both sides of the equation the square of one-half the coefficient of x.

(4) Extract the square root of both sides of the equation and then transpose all but x to the right-hand side.

Example 1:

The depth of a building lot is 10 ft. more than twice its width, and the area is 5500 sq. ft. What are the width and depth of the lot?

Let x = the width of the lot.
Then $2x + 10$ = the depth of the lot.
$x(2x+10) = 2x^2+10x$ = the area, or 5500 sq. ft.
$2x^2+10x = 5500$
Divide by 2, $x^2+5x = 2750$
Add $(\frac{5}{2})^2$, $x^2+5x+6.25 = 2756.25$
Extract the square root, $x+2.5 = \pm 52.5$
$x = -2.5 \pm 52.5$
$= +50$ or -55

Evidently 50 is the value of x that fits the problem. Then the width of the lot is 50 ft. and the depth is $2\times50+10$, or 110 ft. This checks with the value given for the area, 5500 sq. ft.

We can now see what the meaning is of the other value of x. If we took the value of x as -55, then the other dimension, $2x+10$, would be $2\times(-55)+10 = -100$. The product of -55×-100 would give the area $+5500$, so that this value of x would fit the equation, but it would have no practical relation to the problem.

Example 2:

A rectangular nameplate for a machine is to be $1\frac{1}{2}$ in. longer than it is wide and to have an area of 10 sq. in. What will be its dimensions?

Let x = the width of the plate,
then $x+1.5$ = the length of the plate,
$x(x+1.5) = 10$, the area,
$x^2+1.5x = 10$

To make the left hand side a perfect square we must add the square of $\frac{1}{2}$ of 1.5, or $.75^2$, or .5625.

$x^2+1.5x+.5625 = 10.5625$
$(x+.75)^2 = 10.5625$
$x+.75 = \sqrt{10.5625} = \pm 3.25$
$x = \pm 3.25 - .75 = +2.5$ or -4

Since a negative measurement in a problem like this would not have any practical connection, we can take the $+2.5$ as the desired width of the plate. The length will then be $1\frac{1}{2}$ in. more than this or $2\frac{1}{2} + 1\frac{1}{2} = 4$ in. This gives the desired area of 10 sq. in., so the dimensions of the plate are $2\frac{1}{2}$ and 4 in.

The other value of x (-4) has no relation to the problem, but it would fulfill the equation satisfactorily. If x were -4, then the other dimension, $x+1.5 = -4+1.5 = -2.5$. The product of -4 and -2.5 would give $+10$ the same as before. These values are therefore mathematically possible, but are not practical for this problem.

PROBLEMS

Find the values of the unknowns in the following simultaneous equations:

81. $x+y=4$
$x-2y=1$

82. $2x+y=9$
$x-4y=0$

83. $5x+4y=23$
$4x+5y=22$

84. $x+y+z=6$
$x-y+z=2$
$x+y-z=4$

85. Find the lengths of the two links a and b in Fig. 32 so that when the links are in the extended position shown at the left the distance between the centers m and n will be 22 in.; and so that when overlapped as shown at the right the distance between m and n will be 11 in.

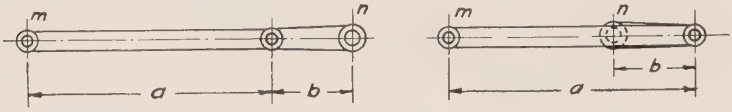


FIG. 32.

86. The total operating cost of an automobile factory is partly fixed and partly dependent on the output. In other words, if A represents the fixed expenses, B the variable expense per machine manufactured, N the number of machines built per year, and C the total operating cost, these items are related as shown in the following equation:

$$A + BN = C$$

One year they built 1000 machines and the total operating cost was \$620,000; the next year they built 1600 machines at a total cost of \$917,000. Calculate the fixed operating cost per year (A) and the variable operating cost, per machine (B).

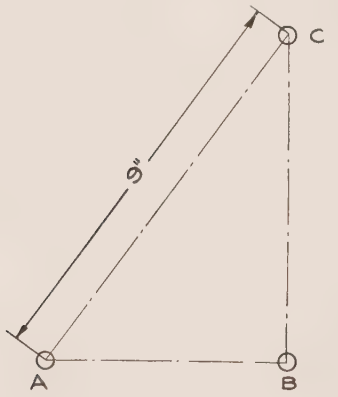


FIG. 33.

87. From the results of problem 86 calculate the total operating cost in a year if 2400 machines were built. Also calculate the total cost per machine in each of the three years.

88. A foundry has a lot of scrap iron which contains 2 per cent silicon and some pig iron which contains 5.2 per cent silicon. The foreman wants to mix these so that there will be 3.35 per cent silicon in the mixture. Calculate the weight of scrap iron and of pig iron that must be used to make each 100 lb. of the mixture.

89. The distance between the centers of two parallel shafts is 7 in. We want to connect these shafts by a pair of gears so that one shaft will turn 1.8 times as fast as the other. Calculate the diameters of each of the two gears.

90. Three holes are to be drilled so that they will lie at the three corners of a right triangle as shown in Fig. 33. The distance from A to C is to be 9 in. and the distance from B to C is to be 2 in. more than that from A to B. Find the distances AB and BC.

CHAPTER VIII

TABLES AND CURVES

37. Use of Tables.—When two quantities are related so that a change in one quantity produces a corresponding change in the other, the relation can be expressed by an algebraic equation. The equation $y=2\ x+3$ states a relation of this sort; whatever value we give to x, there will be a corresponding value of y, which will be 3 more than twice the value of x. If x is 0, y is $2\times0+3$, or 3; if x is 1, y is $2\times1+3$, or 5; if x is 3, y is $2\times3+3$, or 9; if x is 4, y is 11; and so on. Such a relation can be expressed in general terms by the equation or formula, which makes it necessary to perform certain calculations every time we want to find the value of y for a certain value of x. Some of these equations that have applications to shop work we have already used. For example, we know that the equation $W=1\frac{1}{2}\ D+\frac{1}{8}$ in. expresses the relation between the diameter of a bolt and the width of a rough head or nut.

When such a relation is used frequently, we can save time by calculating the corresponding values and arranging them in a table, to avoid calculations at some future period. In this way numerous handbooks have been compiled, containing tables that give such relations as are commonly used in engineering work. In nearly every handbook we will find tables giving the dimensions of bolts of different diameters. In the first column we find the commonly used diameters. Opposite the diameters, in parallel columns, are the other dimensions for these same diameters, as shown in the following extract from a bolt table.

All of these items were originally calculated from formulas, but the formulas have been almost entirely forgotten and have been replaced by the tables. The formulas are now used only for the odd sizes which lie beyond the range of the tables. Sometimes the formulas are given with the tables. When not, they can usually be formulated from an examination of the tables and then applied to any special cases not given in the tables. We have already stated the formula for the width of the bolt-heads and nuts. From the table it will be noticed that the thickness of

DIMENSIONS OF U. S. S. BOLTS AND NUTS

Diameter of bolt	Threads per inch	Diameter at root of thread	Width of flat	Short diameter of heads and nuts, rough	Thickness of heads	Thickness of nuts
$\frac{1}{4}$	20	.185	.0062	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{5}{16}$	18	.240	.0070	$\frac{19}{32}$	$\frac{19}{64}$	$\frac{5}{16}$
$\frac{3}{8}$	16	.294	.0078	$\frac{11}{16}$	$\frac{11}{32}$	$\frac{3}{8}$
$\frac{7}{16}$	14	.344	.0089	$\frac{25}{32}$	$\frac{25}{64}$	$\frac{7}{16}$
$\frac{1}{2}$	13	.400	.0096	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{1}{2}$
$\frac{9}{16}$	12	.454	.0104	$\frac{31}{32}$	$\frac{31}{64}$	$\frac{9}{16}$
$\frac{5}{8}$	11	.507	.0113	$1\frac{1}{16}$	$\frac{17}{32}$	$\frac{5}{8}$
$\frac{3}{4}$	10	.620	.0125	$1\frac{1}{4}$	$\frac{5}{8}$	$\frac{3}{4}$
$\frac{7}{8}$	9	.731	.0140	$1\frac{7}{16}$	$\frac{23}{32}$	$\frac{7}{8}$
1	8	.837	.0156	$1\frac{5}{8}$	$\frac{13}{16}$	1

Note.—The Manufacturers' Standard list, which is also in common use, has dimensions of heads and nuts differing from these given here.

any head is just one-half its width. Then we know that the thickness of the head can be expressed by the formula

$$\text{Thickness of head} = \frac{3}{4}D + \frac{1}{16} \text{ in.}$$

The thickness of the nut is in every case the same as the bolt diameter. Therefore

$$\text{Thickness of nut} = D$$

The number of threads was originally calculated from the bolt diameter by the formula

$$N = \frac{1}{.24\sqrt{D} + .625} - .175$$

taking the nearest whole number to the value of N calculated from the formula.

The diameter at the root of the threads is

$$D' = D - \frac{1.3}{N}$$

while the width of the flat at the top of the thread and bottom of a space is

$$w = \frac{1}{8N}$$

In every shop and drafting room there are opportunities for saving time by preparing tables that will avoid the necessity of making the same calculations repeatedly. For example: lathes are equipped with tables giving the change gears to be used for cutting various threads; milling machines are supplied with indexing tables that give directions for dividing circles into different numbers of equal parts; in many shops a specially prepared table is provided for each lathe indicating the proper speed to run the lathe for turning different sizes of stock and different materials.

38. Use of Curves.—The use of curves on squared paper gives us a method by which it is possible to see at a glance the relation of one quantity to another. These curves may represent the results of calculations from a formula or of observations or experiments in which one quantity depends on the value of another. For this purpose, paper is used having equidistant horizontal and vertical lines, thus dividing the paper into a number of small squares. This paper is variously called *plotting paper, coordinate paper, cross-section paper,* or *squared paper.* Upon this paper we can *plot* the results of our calculations, observations, or experiments and thus get a curve or line which shows the relation between the two quantities.

Suppose that the following observations were taken of the loads on a small electric light plant from 4 p. m. to 12 p. m.

Time, hours p. m.	Load in kilowatts
4	20
5	29
6	50
7	55
8	56
9	47
10	28
11	21
12	19

These observations of time and load can be plotted as shown in Fig. 34. Along the lower line we have laid off the hours at equal distances, and on the vertical line at each hour we have laid off the load according to a scale given along the left-hand edge. In using cross-section paper, remember that each division along

an axis must represent the same unit as every other equal division on that axis. The small circles show the locations of these observations. By drawing a smooth curve through these points we are enabled to see at a glance the variations of the load and to get from the curve the probable load at times in between those given in the observations. For example, at 4:30 p. m. the load was about 23 K. W.; at about 9:18 the load was just 40 K. W. Each small division vertically represents 1 K. W. On the horizontal base line we have used ten divisions for each hour, so each division represents six minutes. Eighteen minutes after the hour would be three divisions to the right of the hour.

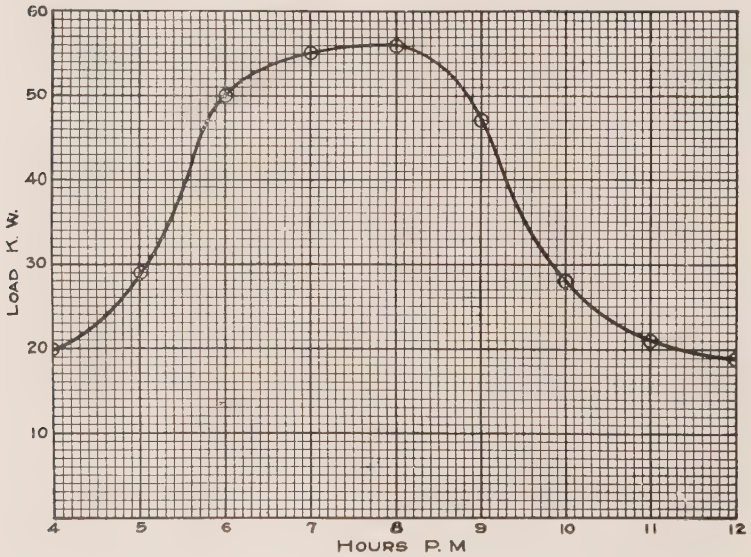


FIG. 34.

It is surprising how much information one can derive from such a curve as this and how much more quickly it shows relations than a table giving the same observations. For instance, we can see what percentage of the people of the town retire between, say, 9 and 10 p. m. By comparing charts for different nights we can tell on what evenings the business places are open; and the season of the year can be readily told by the time at which people begin to use their lights. In the summer months very few lights would be used before 8:30 p. m.

The general name of *curve* is commonly given to the line drawn through the plotted observations, although it may in many

cases be a straight line. Among other terms used are: *graph*, *diagram*, and *chart*.

The quantities that are laid out or plotted horizontally are called *abscissas*.

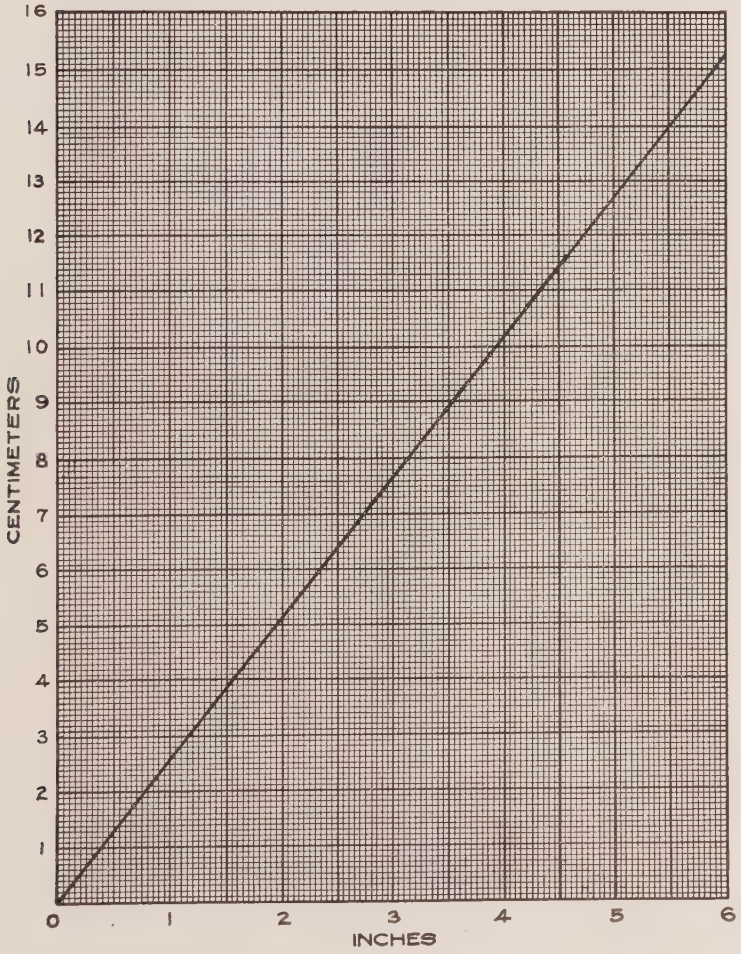


FIG. 35.

The quantities that are plotted vertically are called *ordinates*.

The abscissa and ordinate of any point are together called its *coordinates*.

The operation of laying out the points for a curve from the observations is called *plotting*.

In Fig. 35 is shown the use of plotting paper in plotting the results of certain calculations. In Fig. 35 we have plotted the relation between the centimeter and the inch. The centimeter is a common unit of measure in Europe. There are almost exactly 2.54 cm. in 1 in. The abscissas in this diagram represent inches. The ordinates represent the numbers of centimeters equivalent to the corresponding inches. For instance 1 in. $=2.54$ cm.; 3 in. $=7.62$ cm.; 5 in. $=12.70$ cm., etc. For such a chart to be useful in shop work it would have to be plotted to a large scale and on paper having very fine divisions so the readings could be made accurately. As an example of the use of such a chart suppose we wish to find the metric equivalent of 4 in. Starting at 4 on the horizontal axis, we move vertically till we come to the curve. (In this case the "curve" happens to be a straight line.) Then we move horizontally until the vertical axis is reached and we find that we have between 10.1 and 10.2 cm. If the diagram had been plotted on a larger scale, we would have been able to read it more exactly as 10.16 cm.

A diagram of this sort has one great advantage over a table in that a table gives only certain values, while with a chart we can get intermediate values. A table of the relation between inches and centimeters could give only certain values. With a curve we can get any values within the range of the diagram.

Another use for graphs is in showing the results of experiments or tests. For example, the following data was obtained from loading a helical spring made of $\frac{3}{8}$ in. round steel rod, with 10 coils each 2 in. outside diameter. By measuring the amount that the spring was shortened from its free length by each load we get the following table:

Load in pounds	Deflection in inches
50	.07
100	.15
150	.22
200	.30
250	.36
300	.43
350	.51
400	.58

Using loads as abscissas and the deflections as ordinates in Fig. 36, the observations are located at the small circles in the figure. Now in this case, we do not draw the curve through all of these points, as we did in Fig. 35, because we know that the spring must follow some definite law in deflecting and also that there is a possibility of the readings of deflection being off slightly one way or another, because they were not measured closer than

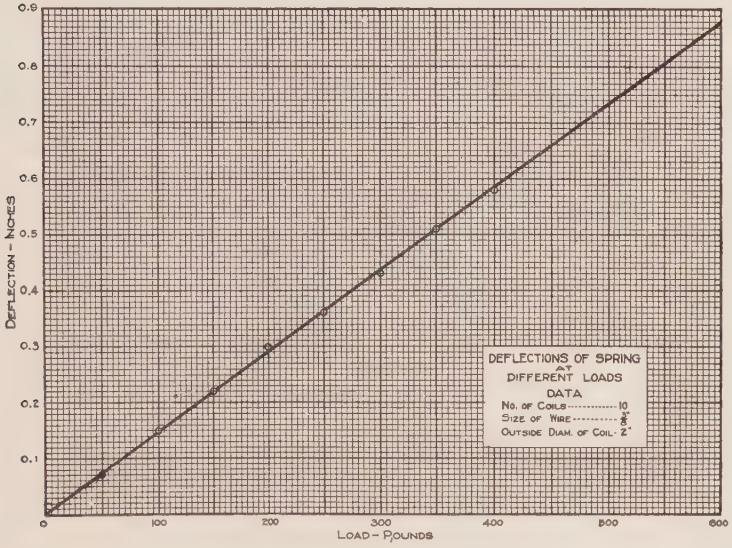


Fig. 36.

.01 in. The points appear to lie so nearly in a straight line that probably the exact relation would be that of a straight line. Therefore, we draw a straight line so that it will average up the readings; that is, so that there will be as many readings on one side of the line as on the other. This line probably indicates the true deflections at the different loads.

Fig. 37 shows another case where the curve should not be drawn through all the points plotted but, instead, a smooth curve is drawn which passes as near as possible to all the points. This curve shows the efficiency of a 250 H. P. gas engine at different loads. The different loads at which the engine was tested are used as abscissas and are given in terms of the rated horse-power of the engine. The ordinates represent the percentage of the energy of the gas that is delivered by the engine as mechanical

work at the crank-shaft. In this case the efficiency calculations may be in error more or less and it is natural to suppose that the efficiency will not jump from point to point but will increase regularly along a smooth curve as the load is increased. At a zero load the efficiency is zero because all the fuel used is expended in overcoming the friction of the engine and only enough gas is used to keep the engine running without any load.

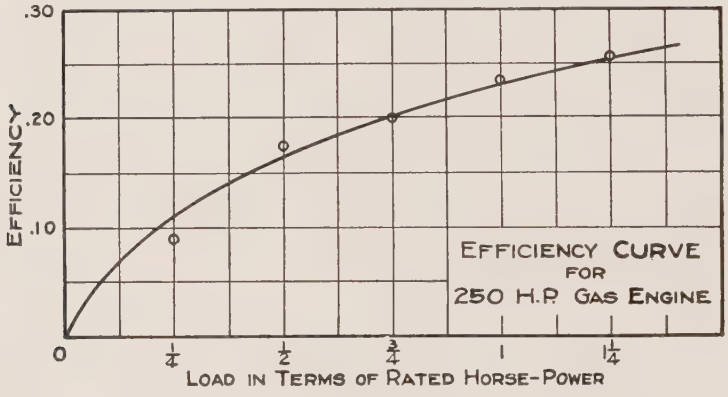


FIG. 37.

PROBLEMS

91. In Fig. 34, what was the load on the machine at the following hours: (a) 4 : 30; (b) 7 : 30; (c) 9 : 30; (d) 10 : 15?

92. At what two times was the load in Fig. 34 at 35 K. W.?

93. From Fig. 35, read as closely as you can the number of centimeters equal to the 6 in. and compare it with the number calculated from the ratio given on page 58. Calculate what part of an inch your reading was in error.

94. A formula often used for helical steel springs is

$$F = \frac{NW(D-d)^3}{1{,}500{,}000\ d^4}$$

in which F is the deflection in inches,

N is the number of coils,

W is the load on the spring,

D is the outside diameter of the coils, and

d is the diameter of the rod or wire from which the spring is made.

Using this formula and the data given with Fig. 36, calculate the deflection that would be produced by a load of 500 lb. on this spring, and compare with the deflection as shown by the diagram.

95. A certain manufacturer quotes the following prices on gasoline engines of different horse-powers: 1½ H. P., $40; 2½ H. P., $60; 4 H. P.,

$100; 5 H. P., $120; 7½ H. P., $175; 10 H. P., $225; 15 H. P., $350. Make a diagram of these prices using horse-powers as abscissas and prices in dollars as ordinates.

96. From your diagram for problem 95, determine what would be the probable price of engines of 6 and 12 H. P. Then, by extending the curves slightly to the right in the same general direction that it has at 15 H. P., find the probable price of an engine of 20 H. P.

97. The pressure in pounds per square inch produced by storing water in a tank or standpipe under a head of H ft. is given by the formula:

$$P = .434\ H.$$

Using heads in feet as abscissas and pressures in pounds per square inches as ordinates, make a diagram showing the corresponding pressures for heads up to 100 ft. Calculate the pressures for every 10 ft., plot them and draw the curve through them.

98. The cutting speed of the work in a lathe is the product of the circumference of the stock, in *feet*, times the revolutions per minute. Prepare a table to give the necessary revolutions per minute for different diameters of stock to produce cutting speeds of 20, 40, 60, 80, and 100 ft. per minute. Calculate for all diameters from 1 in. to 12 in. inclusive, by inches. Arrange the columns as shown in the sample heading below. Before calculating, write out the formula, transpose it to the desired form, and simplify it as much as possible.

TABLE OF CUTTING SPEEDS

Diameter of stock; inches	Revolutions per minute for cutting speeds of				
	20 ft. per min.	40 ft. per min.	60 ft. per min.	80 ft. per min.	100 ft. per min.

CHAPTER IX

EQUATIONS OF CURVES

39. Curves and Equations.—Since a curve or diagram represents the relation between two different quantities, it naturally follows that every equation can be represented by such a diagram and that in many cases an equation can be derived for a given curve, expressing the same relation as that shown by the curve. Consequently, any formula can be expressed by a curve on plotting paper. We have already, in Chapter VIII, plotted curves to represent certain formulas. The nature of the relation as shown in the formula governs the shape of the curve. Some formulas give straight lines, while others give curves of various shapes. The formula which represents the same relation as a curve is called the *equation of the curve.* In Fig. 35 we have a straight line representing the relation between inches and centimeters. This relation can also be expressed by the equations

$$\text{Centimeters} = 2.54 \times \text{inches}$$

or

$$\text{Inches} = .3937 \times \text{centimeters.}$$

If we let y represent centimeters and x inches, we have $y = 2.54\ x$ as the equation of the line in Fig. 35. In problem 97 the equation $P = .434\ H$ was plotted and also found to be represented by a straight line. In general it will be found that when the unknown in one side of the equation is of the first power and the other unknown in the other side is also of the first power the curve will turn out to be a straight line.

40. Positive and Negative Co-ordinates.—In speaking of the two base lines or reference lines of a curve, it is quite common to speak of the horizontal base as the *x axis* and the vertical base line as the *y axis.* This is because in most equations, unless there is some reason for using other letters, we generally use y and x to represent the two unknown quantities. In plotting such equations it is customary to lay off the values of x as abscissas and the values of y as ordinates. Hence the axis of abscissas is called the x axis and the axis of ordinates is called the y axis.

In discussing the plotting of equations, when we speak of the x and y axes or values of x and y, it will be understood that we are merely using x as a general name for abscissas and y as a name for ordinates. In any particular equation or curve any other letters might be used if they had a greater significance.

In plotting points, we have been plotting values of x to the right from the intersection of the two axes and have been plotting values of y upward from the x axis. If negative values of x or y must be plotted, we can plot negative values of x to the *left* of the y axis and negative values of y *downward* from the x axis. Then, distances measured upward or to the right from the axes will be positive, while distances measured downward or to the left will be negative. In plotting such an equation as

$$y=\frac{1}{2}\,x-\frac{1}{4}\text{ in.}$$

we would assign certain values to x and get the corresponding values of y for plotting.

When $x=$	0	1	2	3	4
$y=$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$1\frac{1}{4}$	$1\frac{3}{4}$

When we plot these values, as shown in Fig. 38, we have a negative value of y $(-\frac{1}{4})$ when $x=0$. When $x=0$ there is a zero

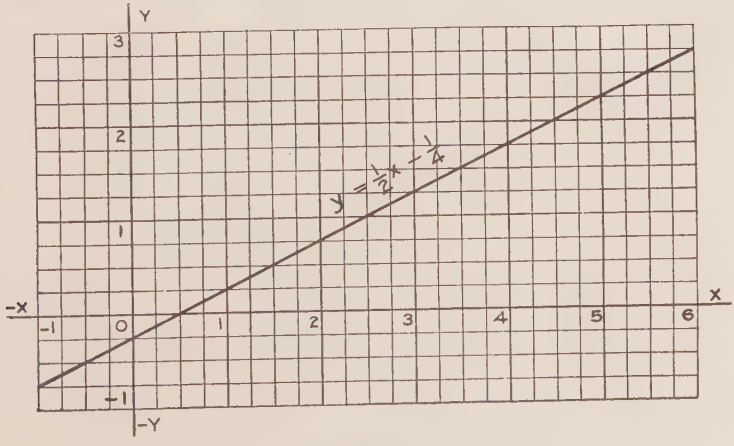


Fig. 38.

abscissa, meaning that the point is located on the y axis, and since this value of y is $(-\frac{1}{4})$ we lay it downward along the y axis. The other values of x and y are positive, and when we plot the points we find that the equation gives the straight line shown in Fig. 38

Taking another equation

$$y = 2 - x$$

we get the following corresponding values for x and y:

When $x =$	0	1	2	3	4
$y =$	2	1	0	−1	−2

Plotting these values, as shown in Fig. 39, we get a line crossing the x axis at $+2$, since $y = 0$ at this point. For values of x greater

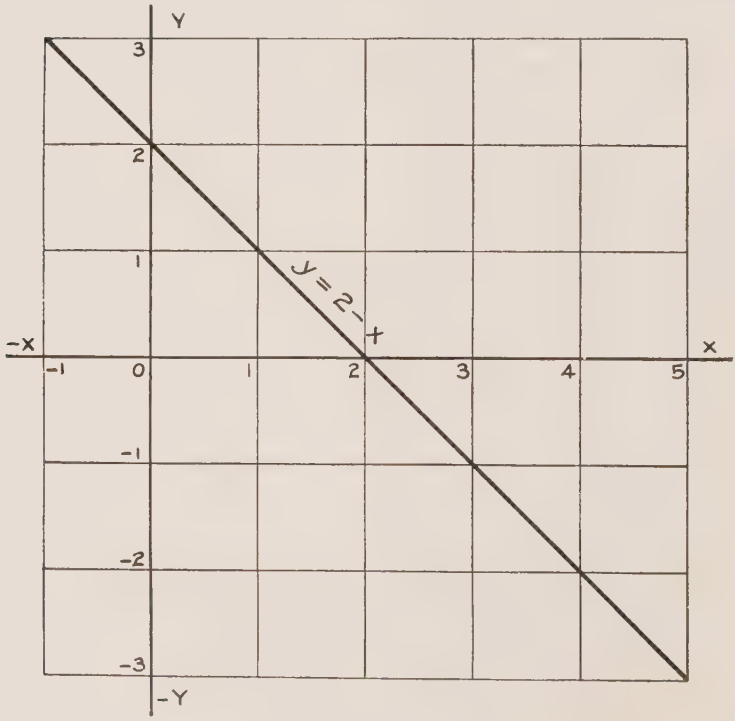


Fig. 39.

than 2, the line is below the x axis, because the values of y are negative when x is greater than 2.

41. Equation of a Straight Line.—In Fig. 40 are shown two lines representing a common rule for keys for shafts. Practice differs among different concerns, but this rule is probably the one in most general use. By it, the width of the key is made equal to one-fourth of the shaft diameter, while the thickness of the key is made one-sixth of the shaft diameter. The two lines on the diagram, therefore, represent the equations

$$w = \frac{1}{4} D, \text{ and, } t = \frac{1}{6} D.$$

In plotting these equations we would assume certain values for D and find the corresponding values of w and t.

When $D =$	0	1.	2.	3.	4.	5.	6.
$w =$	0	.25	.5	.75	1.	1.25	1.5
$t =$	0	.167	.33	.5	.667	.833	1.

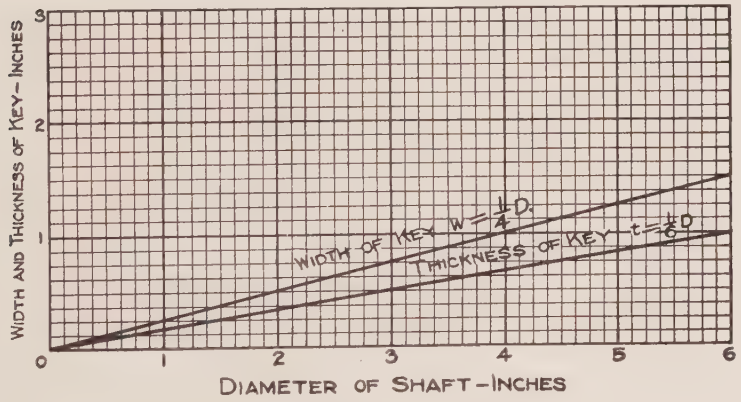


FIG. 40.

By laying up these values of w as ordinates with the corresponding values of D as abscissas and then drawing a line through these points, we get the upper line. In the same manner, using the values of t, we get the lower line.

In using the diagram to get the dimensions of a key for a shaft,

we would use the nearest common fraction to a dimension shown by the diagram. For a $3\frac{1}{2}$-in. shaft the width is just $\frac{7}{8}$ in. The thickness, as seen on the chart, is a little under $\frac{5}{8}$ in. (the formula gives .583 in.) so we would use $\frac{5}{8}$ in. as the thickness.

These two lines are very similar except that one slopes more than the other. The equations are also similar, the only point of difference being the coefficient of D. This at once suggests a possible relation between the slope of the lines and the coefficient of D. Again examining the curves, we find that the line for the width of the key rises 1 in. for each 4 in. of diameter of the shaft; in other words it slopes upward toward the right 1 in 4, or $\frac{1}{4}$. Likewise the line for the thickness slopes upward 1 in 6, or $\frac{1}{6}$. These fractions representing the slopes of the lines are the same as the coefficients of D. In other words, the coefficient of x (in this case x is D) gives the slope of the line.

In Fig. 38 the coefficient of x is $\frac{1}{2}$ and the line is seen to rise $\frac{1}{2}$ for each increase of 1 in the value of x. Hence its slope is $\frac{1}{2}$, the same as the coefficient of x.

Fig. 39 shows a slightly different situation. Here the coefficient of x is -1 and the line slopes downward toward the right. On this line the value of y decreases 1, or changes by -1, for a change of $+1$ in the value of x. The line therefore slopes -1 to 1 or $-\frac{1}{1}=-1$. The slope of the line is here also the same as the coefficient of x.

There is another interesting point about these diagrams. In Fig. 38 the equation is $y=\frac{1}{2}\,x-\frac{1}{4}$ and it will be noticed that the y axis is cut by the line at $-\frac{1}{4}$. In Fig. 39 the equation is $y=2-x$ and the y axis is cut by the line at $+2$. In Fig. 40 we have the equations $y=\frac{1}{4}\,x$ and $y=\frac{1}{6}\,x$. In both cases the line cuts the y axis at 0. From these illustrations we notice that the point where the line cuts the y axis is the same as the constant number on the right side of the equation.

In general, then, if any equation is of the form

$$y=ax+b$$

the coefficient a of x is the slope of the line, and b is the point where the line cuts the y axis. If b is zero, the line passes through the intersection of the two axes (see Fig. 40).

If any equation can be reduced to the form of $y=ax+b$, it can be assumed that its curve is a straight line and it is only necessary to locate two points in order to draw the curve with a ruler.

42. Finding the Equation for a Line.—From the preceding article it follows that when we plot a series of observations and get a straight line, it is possible to find an equation for that line which will express the same relation between the quantities as is shown by the line.

Let us suppose that the testing department has made tests on a number of steels containing different percentages of carbon, in order to find the effect of carbon on the strength of steel, and that the following results are obtained:

Percentage of carbon in the steel	Breaking strength in pounds per square inch
.09	53,000
.16	64,000
.20	65,600
.31	77,000
.39	90,000
.50	97,000
.57	110,000
.71	124,000
.79	127,000
.89	140,000

If we plot these values, as shown in Fig. 41, we find that the curve which comes nearest to all the points is a straight line. In Fig. 41 this is the solid heavy line. Now let us find a law or formula that will state the effect of the carbon on the strength of the steel as shown by these tests. Since the curve is a straight line, its equation will have the form of

$$y = ax + b$$

where a and b are two numbers whose values are to be determined for this particular line.

If we take the values of x and y at two points some distance apart on the line and substitute these values for x and y in the equation $y = ax + b$, we will get two simultaneous equations for a and b. When $x = .1$, $y = 56{,}000$ and the equation becomes $56{,}000 = .1\,a + b$. When $x = .89$, $y = 140{,}000$ and the equation becomes

$140{,}000 = .89\ a + b$. We now have two simultaneous equations for a and b:

$$140{,}000 = .89\ a + b \qquad (1)$$
$$56{,}000 = .10\ a + b \qquad (2)$$

Subtracting, $84{,}000 = .79\ a$

From which, $a = 84{,}000 \div .79 = 106{,}330.$

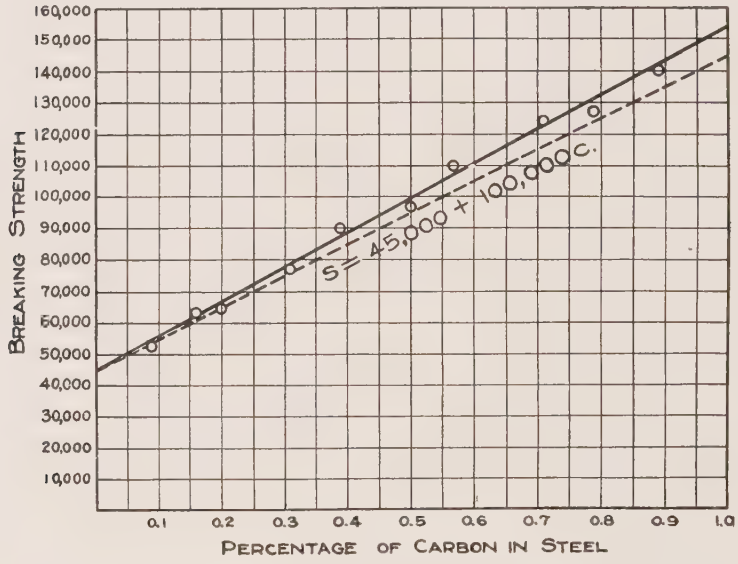


FIG. 41.

We can now substitute this value for a in either equation (1) or (2) and get the value for b, which turns out to be about 45,336. The equation for the line would then be

$$y = 106{,}330\ x + 45{,}336.$$

Since the line cuts the y axis, we could use this to aid in getting the equation in a slightly simpler way. When $x = 0$, y would equal b. At this point, y is about 45,000. Hence b is about 45,000. This gives us at once $y = ax + 45{,}000$ and leaves only a to be found. Taking another point on the curve, such as $y = 140{,}000$ when $x = .89$, we have $140{,}000 = a \times .89 + 45{,}000$, from which $a = 106{,}740$. This gives the equation $y = 106{,}740\ x + 45{,}000$ which is practically identical with that obtained before. Without

materially affecting the line, we can change these figures slightly to make them even numbers that can be more easily used. If we use

$$y = 100{,}000\,x + 45{,}000$$

the equation will represent the dotted line in Fig. 41. This is slightly lower and therefore is on the safe side, in addition to being much simpler. Instead of y and x we can use S for the strength and C for the per cent of carbon, thus making the equation, as shown on the diagram,

$$S = 45{,}000 + 100{,}000\,C.$$

43. Equations of Curves.—If an equation contains a power or root of x or y or the product of x and y, it will give a true curve. In such cases we can generally only construct the curve by the

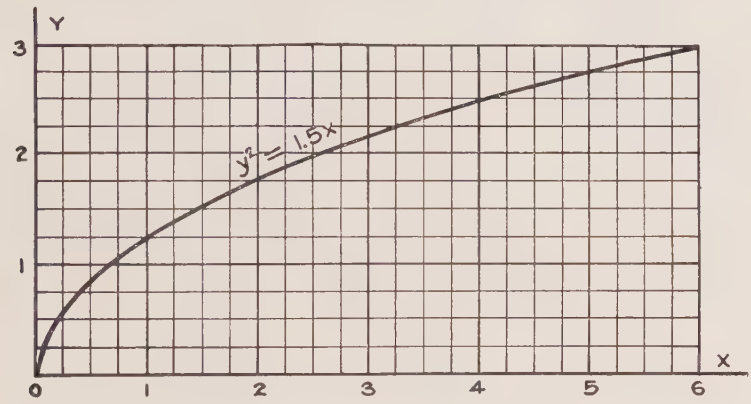


FIG. 42.

process of calculating several values of y and x and drawing a smooth curve through them. Fig. 42 shows the curve plotted from the equation $y^2 = 1.5\,x$, or $y = \sqrt{1.5\,x}$. Here we have assigned certain values to x and calculated the corresponding values of y and then plotted these points and drawn a smooth curve through them. When $x = 0$, $y = 0$; when $x = .5$, $y = \sqrt{.75} = .87$; when $x = 1$, $y = \sqrt{1.5} = 1.24$; when $x = 2$, $y = \sqrt{3} = 1.73$; when $x = 4$, $y = \sqrt{6} = 2.44$; etc. This same equation would also give a similar curve below the x axis because each positive value of x would give both a positive and a negative value of y.

If an equation has the form

$$xy = \text{a constant number}$$

we will get a curve like that in Fig. 43. Curves of this form are often encountered in plotting the relations of two quantities in practical problems. For example, the curve of Fig. 43 shows the R. P. M. at which to run a lathe to give a cutting speed of 50 ft. per minute for different diameters of work. The cutting speed is the product of the circumference of the work, in feet, times the

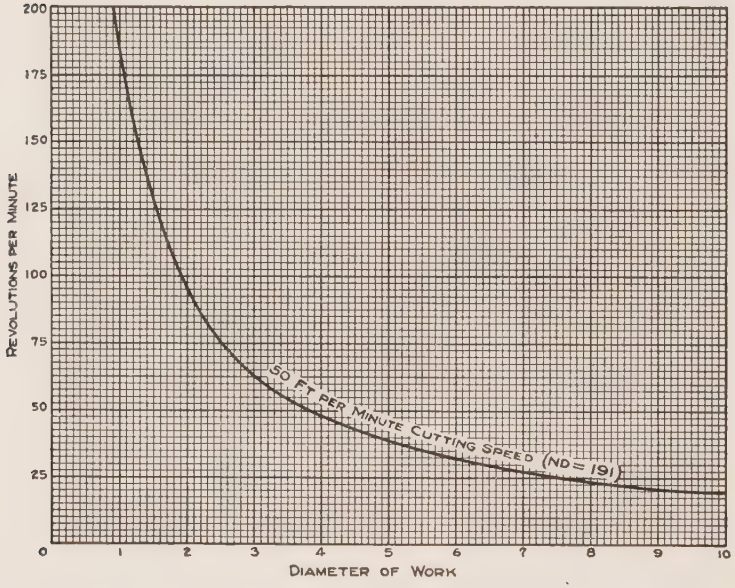


Fig. 43.

revolutions per minute. If D represents the diameter in inches, the circumference in feet will be $\frac{\pi D}{12}$; and if N represents the R. P. M. we will have the following equation for a constant cutting speed of 50 ft. per minute.

$$N \times \frac{\pi D}{12} = 50$$

$$N D = \frac{50 \times 12}{3.1416} = 191$$

Then, if we plot the curve $N\ D = 191$, we can get from it the R. P. M. for work of any given diameter to give a cutting speed of 50 ft. per minute. Other curves might be drawn on the same chart for different cutting speeds. For a speed of 100 ft. per minute the equation would be $N\ D = 382$. With several curves for different speeds, a chart like this would be a valuable thing to have hanging near every lathe or milling machine. It could also have marked along the y axis the R. P. M. of the spindle for the different steps on the cone, with back-gears in and with back-gears out. The machinist could find the diameter of his work along the x axis, run vertically up to the curve of the desired cutting speed, then run horizontally to the y axis where he would find the necessary R. P. M. and the position of the belt and back-gears that would come nearest to giving this speed.

PROBLEMS

Note.—In plotting curves, use scales as large as the sheet will allow, so that the curves can be drawn and read more accurately.

99. From the equation for the line in Fig. 38 calculate at what value of x the line crosses the x axis.

100. From Fig. 40 determine the dimensions of a key for a $4\frac{1}{2}$-in. shaft.

101. From Fig. 43 determine:

(a) What size of stock should be run at 125 R. P. M. to give the cutting speed of 50 ft. per minute.

(b) The proper R. P. M. for a machine steel shaft $4\frac{1}{2}$ in. in dameter to give the same cutting speed.

102. The following formula is recommended by some authorities for allowances for running fits between a shaft and its bearing:

$$a = .001 \text{ in.} + .0008\sqrt{d}$$

where d is the diameter of the bearing in inches and a is the allowance by which the shaft should be made smaller than its bearing. Calculate to five decimal places the allowances for each inch from 0 to 10 in. diameter. Plot a curve to show the allowances using diameters as abscissas and allowances in *thousandths* of an inch as ordinates.

103. The formula adopted by the Society of Automobile Engineers for calculating the horse-power of automobile engines is

$$\text{Horse-power} = \frac{D^2 N}{2.5}$$

where D is the cylinder diameter in inches and N is the number of cylinders on the engine. Construct two curves on the same sheet, one curve for engines with 4 cylinders and one for 6 cylinders. Use diameters up to 6 in. as abscissas. Plot horse-powers as ordinates and select the scale of ordinates so that the curves will at least show the horse-powers of all sizes between 3 in. and 6 in. in diameter

104. A large electrical manufacturing concern charges the following prices for a certain type of motor of different horse-powers: 1 H. P., \$70; 2 H. P., \$90; 3 H. P., \$105; 5 H. P., \$145; $7\frac{1}{2}$ H. P., \$205; 10 H. P., \$255; 15 H. P., \$350. Plot the relation of price to horse-power, using horse-powers as abscissas and prices as ordinates; draw a line to represent as nearly as possible the relation of price to horse-power and find the equation of the line.

CHAPTER X

GEOMETRIC CONSTRUCTIONS

44. Definitions.—Geometry treats of the properties, construction, and measurements of lines, surfaces, and solids. A knowledge of its principles is very useful to shop men in laying out work and in calculating the areas of figures and the volumes and weights of objects.

A *Solid* has three dimensions, usually designated as *length*, *breadth*, and *thickness*.

A *Surface* has but two dimensions—*length* and *breadth*.

A Line has but one dimension—*length*. In actually drawing a line with a pencil or scribe, it must be given some width in order to be visible. In theory, however, a line has no width.

A *Point* has no dimension but is simply a position in space. In marking a point with a pencil, prick, punch, or scribe, it must of necessity, however, be given some size so as to be visible to the eye. In locating a point on an object, three dimensions are necessary to determine its position. On a surface, a point is located by two dimensions, usually given as the distances from two boundary lines or from the center lines, or it may be located by giving its direction and distance from some other point. On a

Fig. 44.
Parallel Lines.

Fig. 45.
Horizontal Line.

Fig. 46.
Vertical Line.

line, a point is located by a single dimension, either the distance from one end of the line or from some other known point on the line.

Parallel Lines are lines that are everywhere equally distant from each other, as *AB* and *CD* in Fig. 44.

A *Horizontal Line* is a line parallel to the horizon, or water

level. The surface of water in a tank is horizontal. *EF* in Fig. 45 is a horizontal line.

A *Vertical Line* is a line straight up and down. *GH* in Fig. 46 is a vertical line.

A line is *perpendicular* to another when it meets or crosses the other line squarely, as shown by the lines *AB* and *CD* in Fig. 47. These lines are also said to meet or cross *at right angles.* A line is also said to be *normal* to another when it meets the other at right angles.

45. Angles.—An Angle is formed by the meeting or intersection of two straight lines. The intersection of the lines *AB* and *CD* in Fig. 48 gives four angles, *AOC*, *AOD*, *DOB*, and *BOC*. An angle is measured by the difference in direction of the two

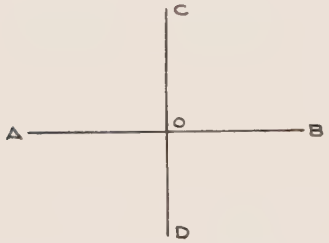


Fig. 47.

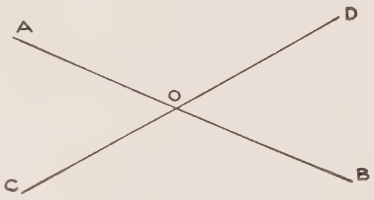


Fig. 48.

lines which form the angle. The point of the angle where the lines meet is called the *vertex.*

The sign ∠ is often used for the word *Angle.*

Thus ∠*AOD* is read "*the Angle AOD.*" The middle letter is always the letter at the vertex.

A *Right Angle* is the angle formed by two lines that are perpendicular to each other. In Fig. 47 each of the angles *BOC*, *COA*, *AOD*, and *BOD* is a right angle.

An angle less than a right angle is called an *acute* angle. Thus ∠*BOD* in Fig. 48 is an acute angle.

An angle greater than a right angle is called an *obtuse* angle. Thus ∠ *DOA* in Fig. 48 is an obtuse angle.

46. Circles and Circular Measure.—A Circle is a flat surface bounded by a curved line all points of which are equally distant from a point within called the *Center.*

The bounding line of a circle is called the *Circumference*, as shown in Fig. 49.

A straight line drawn from the center to the circumference is called the *Radius*.

A straight line through the center, terminating at both ends on the circumference, is called the *Diameter*. The diameter of a circle is equal to twice the radius. The circumference of a circle is equal to 3.1416 times the diameter.

An *Arc* of a circle is any portion of a circumference, as *AB* in Fig. 50.

A *Chord* of a circle is a straight line connecting any two points on the circumference, as the line *CD* in Fig. 51.

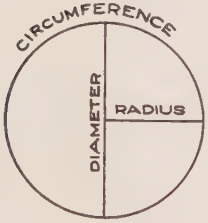


Fig. 49.

Fig. 50.

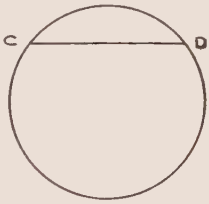


Fig. 51.

Circular Arcs are measured in *degrees*, there being 360 degrees in a complete circumference. For finer measurements, degrees are divided into *minutes*, and minutes into *seconds*, there being 60 seconds in 1 minute and 60 minutes in 1 degree. Degrees are marked by the sign (°), minutes by (′), and seconds by (″).

TABLE OF CIRCULAR MEASURE

60″ (seconds) = 1′
60′ (minutes) = 1°
360° (degrees) = 1 circumference.

47. Angular Measure.—Angles also are measured in degrees, minutes, and seconds, by swinging an arc of a circle from one side of the angle to the other, using the vertex of the angle as the center. The number of degrees in the arc gives the number of degrees in the angle. A right angle therefore contains 90°, because, as shown in Fig. 52, two lines at right angles form four right angles, and these divide a complete circumference of 360° into four equal parts of 90° each.

An acute angle has less than 90° while an obtuse angle contains more than 90°.

The *Complement* of an angle is the difference between that angle

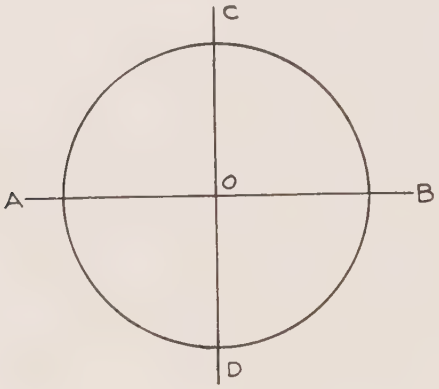


Fig. 52.

and 90°. Hence, two angles are said to be *complementary angles* when their sum is 90° or a right angle.

The complement of $60° = 90° - 60° = 30°$

The complement of $20° = 90° - 20° = 70°$

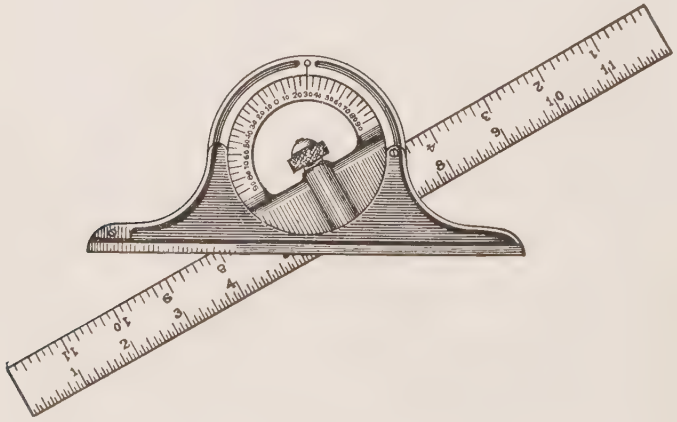


Fig. 53.

The *Supplement* of an angle is the difference between that angle and 180°. When two supplementary angles have a common

vertex and have one side in common, their other sides form a straight line. For example, in Fig. 48 the angles AOC and AOD are supplementary angles.

The supplement of $60° = 180° - 60° = 120°$
The supplement of $35° = 180° - 35° = 145°$

In laying out angles in shop work, a tool called a *protractor* is used. The protractor shown in Fig. 53 is formed by attaching a steel rule to a protractor head. The inner circle, carrying the scale, can be swung to make any desired angle with the face of the head. The circle is graduated to show the number of degrees in the angle between the scale and the face of the head.

There are other ways of laying out or measuring angles without the use of a protractor. One convenient method is to measure the opening of a 2-ft. rule as shown in Fig. 54. By knowing the

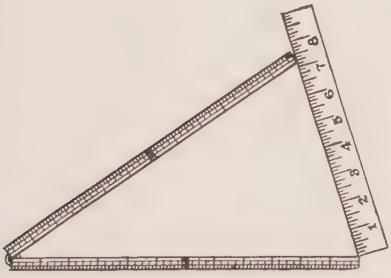


Fig. 54.

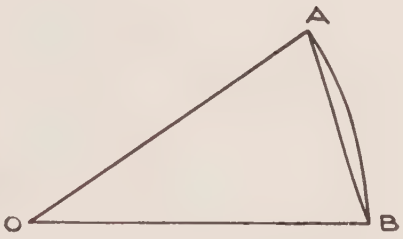


Fig. 55.

proper distance between the open ends of the rule for a certain angle we can set the ends of the rule to that distance and scribe the angle along the edges of the rule. In Fig. 55 the angle AOB is measured by the number of degrees in the arc AB. If we draw the straight line AB, it will be seen to be the chord of the arc AB. With any given radius and given arc, this chord will have a definite value. To lay out angles by the 2-ft. rule, as in Fig. 54, we must know the lengths of the chords at a radius of 12 in. for the different angles. The following table from the American Machinist's Handbook gives the chords for a 12-in. radius to be used with a 2-ft. rule. The same table can be used in measuring angles if we measure the chord of the angle at a radius of 12 in.

DEGREES AND OPENING OF A 2-FT. RULE

Degrees	Inches	Degrees	Inches	Degrees	Inches
1	.21	15	3.12	55	11.02
2	.422	20	4.17	60	12.
3	.633	25	5.21	65	12.87
4	.837	30	6.31	70	13.76
5	1.04	35	7.20	75	14.61
7.5	1.57	40	8.21	80	15.43
10	2.09	45	9.20	85	16.21
14.5	3.015	50	10.12	90	16.97

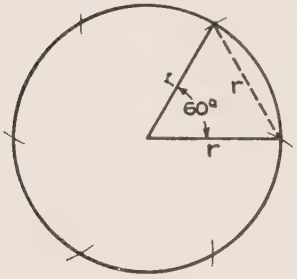


Fig. 56.

An interesting and valuable fact to know in laying out angles is that a chord equal to the radius of a circle will lay off an arc of exactly 60° and, hence, will mark off an angle of just 60° at the center of the circle, as shown in Fig. 56. Since 60° is just one-sixth of 360°, it will be seen that a circle can be divided into six equal parts by stepping around it with the dividers set to the radius of the circle.

48. To Draw a Line Parallel to Another Line and at a Given Distance from It.—Let AB, Fig. 57, be the given line. Set the compass or dividers to the desired distance between the two lines and, with any two points on AB such as m and n as centers, scribe the arcs xy and $x'y'$. Draw the upper line CD tangent

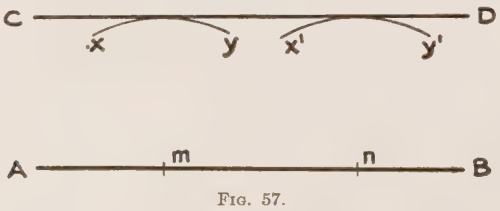


Fig. 57.

to (just touching) these two arcs. CD will be parallel to AB and will be the desired distance from it.

49. To Bisect a Line (divide it in two equal parts).—Let AB, Fig. 58 be a line of which we want to find the exact center or

which, in other words, we want to bisect. Set the compass to a radius somewhat more than one-half the length of the line AB. With A and B as centers strike two arcs, as shown, so that they intersect (cross) above and below AB. Through these intersections draw a straight line CD. This line bisects AB and is also

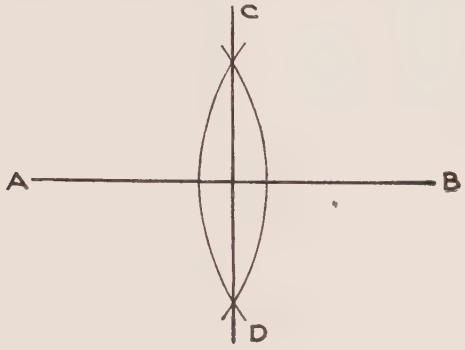


Fig. 58.

perpendicular to it. Hence, CD may be called a *perpendicular bisector* of AB.

50. To Divide a Line into Any Number of Equal Parts.—Let AB, Fig. 59 be a line which we want to divide into, say, three equal parts. From B draw a line BD at any convenient angle with AB. Place a scale or ruler along BD and mark three equal

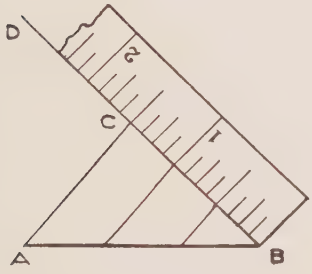


Fig. 59.

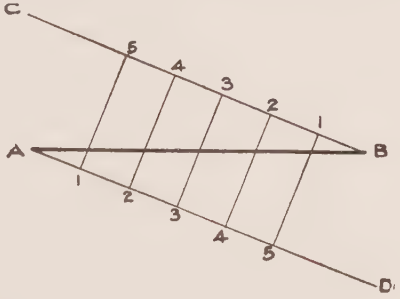


Fig. 60.

spaces along the ruler. In the figure, three $\frac{1}{2}$-in. spaces are used. This gives us three equal spaces on the length BC. Connect C to A by the line CA. From the other two points on BC draw lines parallel to CA. These lines divide AB into three equal spaces.

Another method is shown in Fig. 60, where the line AB has been

divided into six equal parts. From B draw BC making any angle with AB, and from A draw AD making the *same* angle with AB but on the other side of AB. AD and BC are parallel, since they make the same angle with AB. From B measure off with the scale on BC a number of equal distances one less than the number into which AB is to be divided. From A lay off on AD the same distances. Then connect point 1 on AD with 5 on BC. Connect the other points in the same manner, as shown in the figure. These lines are parallel and equidistant and therefore will divide AB into the desired number of equal parts.

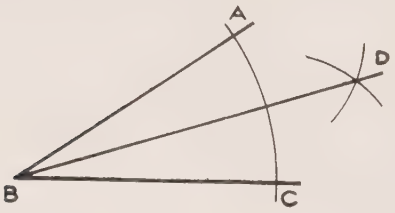


FIG. 61.

51. To Bisect an Angle.—Let ABC of Fig. 61 be an angle that is to be bisected (divided into two equal angles). From B as a center draw any convenient arc cutting the sides of the angle at A and C. Then, using the same radius or any more convenient radius, from A and C as centers describe two intersecting arcs having equal radii. From D, their intersection, draw a straight line to B. The line BD bisects the angle ABC because it is equally distant from AB and BC.

52. To Erect a Perpendicular at Any Point on a Line.—We have already explained (Art. 49) how to draw a perpendicular bisector to a line. In many cases it is necessary to locate a point directly above a given point on a line. In such a case the surest way is to erect a perpendicular to the line at the given point. Let AB, Fig. 62, be a line on which we want to erect a perpendicular at the point C. From any point such as O above the line, scribe a circle of such a radius that it will pass through C. It will also cut the line AB at some other point D. From D draw a line through the center O. It will cut the other side of the circle at some point E. A line drawn from E to C will be perpendicular to the line AB at C.

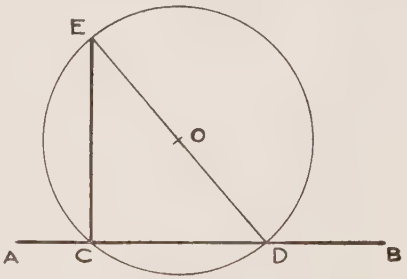


FIG. 62.

Another method is based on the fact that a chord equal to the radius of a circle lays off an angle of 60°. In Fig. 63 let AB be a line upon which to erect a perpendicular at C. About C as center describe an arc DEF. Using the same radius, mark the point E from D thus making DE an arc of 60°. Then from E

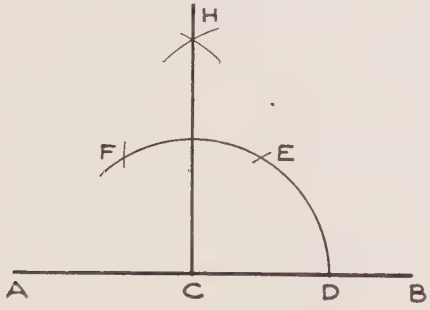


Fig. 63.

mark the point F with the same radius. From F and E swing two arcs of equal radii intersecting at some point H. The line from H to C is the desired perpendicular, since the angle BCH has been constructed equal to $60° + \frac{1}{2}$ of $60° = 90°$.

Another very simple method of constructing a right angle is shown in Fig. 64. This depends on the fact that a triangle having

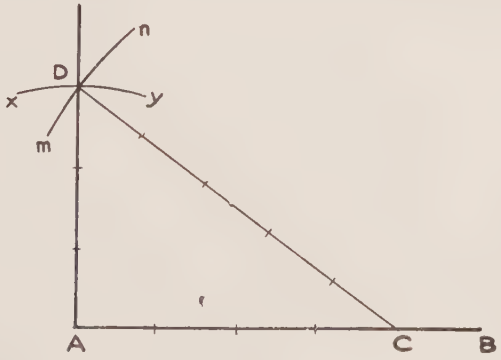


Fig. 64.

its sides in the ratios 3 : 4 : 5 will be a right triangle, because $3^2+4^2=5^2$. Suppose we wish to erect a perpendicular at the point A on the line AB. Select some convenient unit, say $\frac{1}{2}$ in., and mark off four of these units from A along AB, thus locating C. From C swing an arc mn with a radius of five units and from A

swing an arc xy with a radius of three units. Through the intersection, D, of these arcs draw a straight line to A. The lengths AD, AC, and CD will be proportional to $3 : 4 : 5$ and, therefore, the triangle ACD is a right triangle and the angle CAD is a right angle.

53. To Erect a Perpendicular to a Line from a Point not on the Line.—Let AB in Fig. 65 be the given line and C the given point. With C as center describe an arc cutting AB at two points, D and E. With D and E as centers and any convenient radius draw two arcs intersecting at some point H. The line CH will be the desired perpendicular.

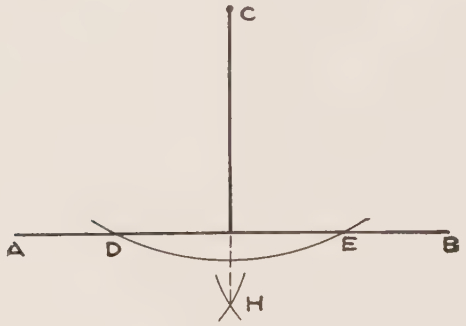


Fig. 65.

54. Construction of Some Common Angles.—The methods of constructing angles of 60° and 90° have already been explained. An angle of 30° can be constructed by bisecting a 60° angle; an angle of 15° can be obtained by bisecting an angle of 30°; and so on. Likewise, by bisecting a right angle we can get an angle of 45° and, by bisecting that, an angle of $22\frac{1}{2}°$.

55. Construction of Equal Angles.—Sometimes we have an angle already constructed and we want to construct its equal or duplicate on another piece. Let ABC, Fig. 66, be the angle that we have given and let ED be another line on which we want to construct an equal angle at the point E. With the vertex B of the given angle as center and with any convenient radius, swing an arc GF cutting both sides of the angle. From E as center, with the same radius, scribe a similar arc HK. Now set the compass or dividers to the distance from G to F, where the first arc cuts AB and BC. Using this as radius and with K as center, scribe the arc xy. Through the intersection H of HK and xy draw the

straight line EH. By this construction we have made the chords HK and FG equal and also the arcs HK and FG equal. Since these arcs are of the same radius and are swung from the vertices B and E, the angles must be equal.

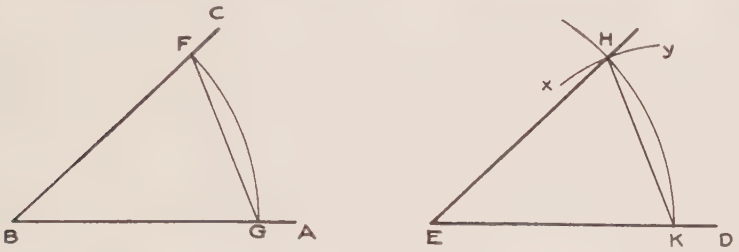


Fig. 66.

56. To Find the Center of An Arc or a Circle.—Select three points, A, B, and C, Fig. 67, well apart on the circumference. With A and B as centers, draw two arcs of the same radius intersecting at some points E and F. Then with B and C as centers draw arcs of equal radius intersecting at some points G and H. Through the intersections E and F and G and H of these pairs of arcs draw the lines EF and GH. The point O where these lines intersect is the center of the circle because a line through the center of a circle is always normal (perpendicular) to the circumference at the point where it crosses the circumference, and we have drawn EF and GH normal to the circumference.

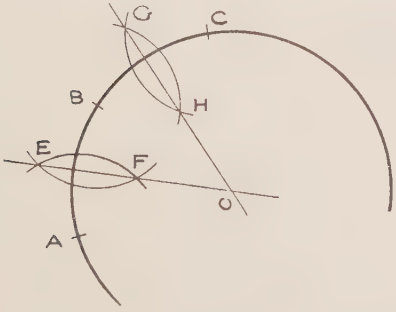


Fig. 67.

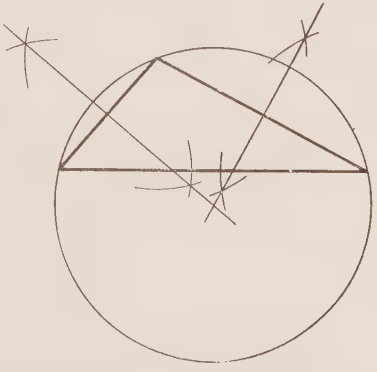


Fig. 68.

57. To Pass a Circle Through Three Points.—Proceed, as just explained, to locate the center of the circle. The rest will be evident.

The same process naturally applies to passing a circle through the three corners of a triangle, as shown in Fig. 68.

It is sometimes necessary to lay out an arc of a circle where the radius is so great as to make it inconvenient to locate the center of the curve. This often occurs in bending steel beams or in cutting curved shapes from steel plates. In Fig. 69, let AB be

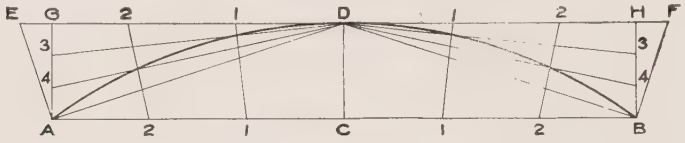


Fig. 69.

the chord of the desired curve and let CD be the rise or spring of the curve in the center. Then the curve is to be passed through A, D, and B. Draw the straight lines AD and BD. Then draw AE and BF perpendicular to AD and DB and draw EDF parallel to ACB. Divide the lines AC, ED, CB, and DF each into the same number of equal parts (in this case three parts have been used). Connect the corresponding points on AB and EF by the

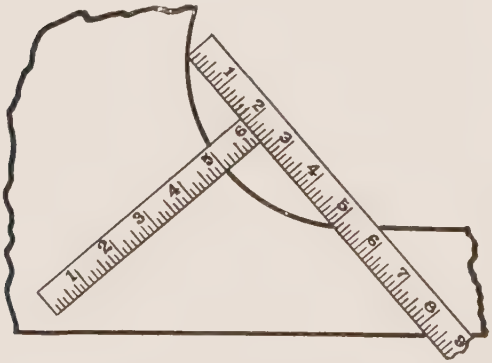


Fig. 70.

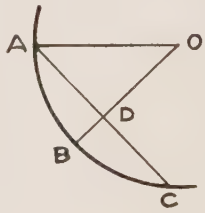


Fig. 71.

lines 1–1 and 2–2. Next, draw the lines AG and BH perpendicular to AB and divide these lines into the same number of equal parts. Draw the lines 3–D and 4–D. The intersections of lines 3–D with 1–1 will give points on the desired curve, as will also the intersections of 4–D with 2–2. The points thus located can then be connected by a smooth curve. A greater number of divisions can be used if greater accuracy is desired.

58. To Find the Radius of an Arc.—It is sometimes necessary to find the radius of an arc where it is not convenient to locate the center of the arc. This often happens in measuring a casting, where we want to find the radius of a fillet or other curve. Fig. 70 shows how this can be done. Lay a scale across the curve to form a chord. Then, with another scale, measure the distance from the center of this chord to the curve. The formula for calculating the radius can be calculated from Fig. 71. If O is the center of the curve, AO and BO are both radii of the curve. From the measurements, we know the distances AD and BD.

Let r = the unknown radius, then

$$DO = r - BD$$

Also, $DO^2 = AO^2 - AD^2 = r^2 - AD^2$

Hence, $(r - BD)^2 = r^2 - AD^2$

$$r^2 - 2r \times BD + BD^2 = r^2 - AD^2$$

$$2r \times BD = AD^2 + BD^2$$

$$r = \frac{AD^2 + BD^2}{2 \times BD}$$

But $AD = \frac{1}{2}$ chord, and BD = height

$$\text{Hence, radius} = \frac{(\frac{1}{2}\text{ chord})^2 + \text{height}^2}{2 \times \text{height}}$$

In Fig. 70 the chord is 5 in. and the height is 1 in.

$$\text{Then, radius} = \frac{(\frac{1}{2}\text{ of } 5)^2 + 1^2}{2 \times 1}$$

$$= \frac{(2\frac{1}{2})^2 + 1^2}{2} = \frac{7\frac{1}{4}}{2} = 3\frac{5}{8}\text{ in.}$$

59. An Angle with its Vertex on the Circumference of a Circle.—It has been previously explained that an angle at the center of a circle is measured by the arc included between the sides of the angle. If, however, the vertex of an angle is on the circumference, the angle is measured by *one-half* of the included arc. In Fig. 72 the angle CAB in degrees is just one-half of the number of degrees in the arc CB. Likewise, the angle DEF is measured by one-half of the arc DF. One particular application of this

is often seen in the shop. From Fig. 73 it will be seen that any angle having its vertex on the circumference and having the sides of the angle leading to the extremities of a diameter is a right angle. The arc included by the sides is a semicircle, or 180°. Hence, the angle is $\frac{1}{2}$ of 180° or 90°, or a right angle. Pattern-

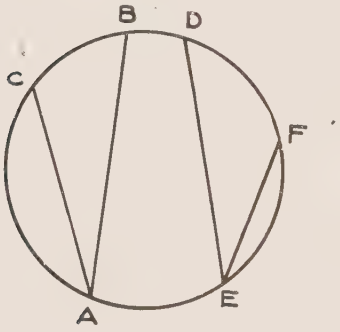


Fig. 72.

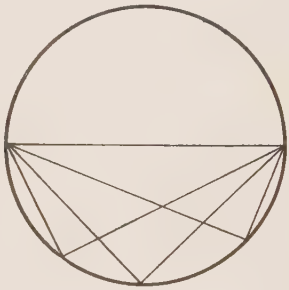


Fig. 73.

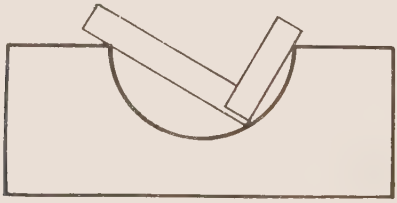


Fig. 74.

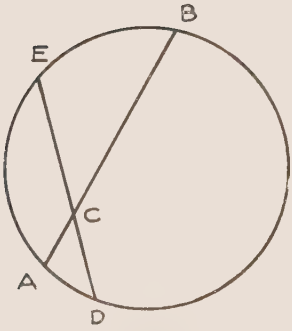


Fig. 75.

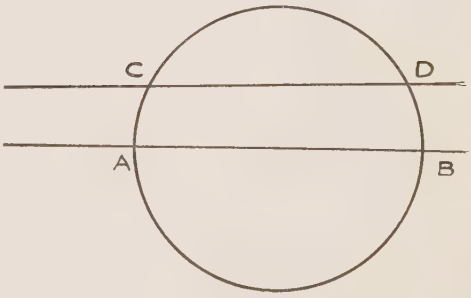


Fig. 76.

makers often make use of this fact in making a half-round groove, for example, in making a core box. If the groove is a true semicircle, the corner of a square will touch the groove at any point when the sides of the square are resting on the edges of the groove, as shown in Fig. 74.

60. Angles with the Vertex Inside or Outside of a Circle.—If two straight lines cross *inside* of a circle, the angle between the lines is measured by one-half the *sum* of the included arcs. Thus, in Fig. 75 the angle ECB or the angle ACD is measured by $\frac{1}{2}$ of (arc EB + arc AD).

If two lines intersect *outside* a circle, the angle between them is measured by one-half the *difference* between the included arcs. This fact can be used to lay out lines having a very slight slope with each other. Thus, in Fig. 76, if the arc AC is 30° while the arc BD is 32°, the angle between the lines AB and CD is $\frac{1}{2}$ of $(32° - 30°) = 1°$.

PROBLEMS

Note.—For the work of this and several following chapters the student will need a ruler or scale and a compass such as can be purchased for from ten to twenty-five cents.

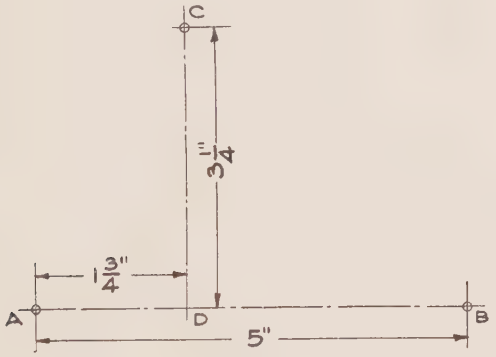


FIG. 77.

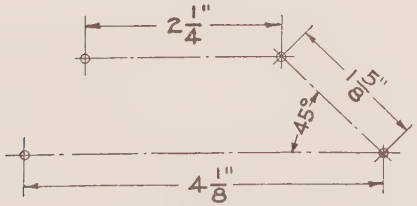


FIG. 78.

105. A pulley has six arms. What angle is included between the center lines of any two adjacent arms?

106. A gear is $12\frac{1}{2}$ in. outside diameter and has 48 teeth. What is the length in inches of the arc from one tooth to the next and what is the number of degrees in the arc?

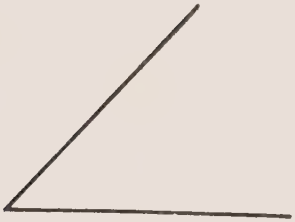


Fig 79.

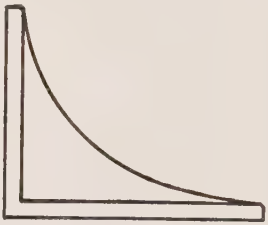


Fig. 80.

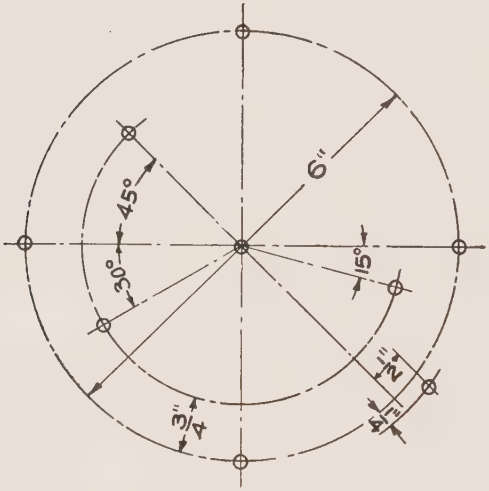


Fig. 81.

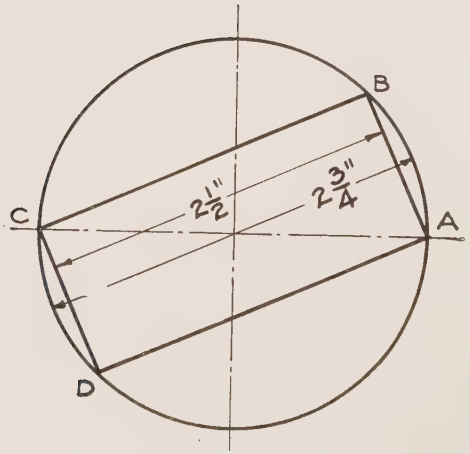


Fig. 82.

107. A belt is in contact with a 36-in. pulley for 187° of the circumference. What is the length in inches of the arc of contact?

108. Three holes are to be laid out on a plate in the positions shown in Fig. 77. Lay these out on paper by erecting a perpendicular at D. Test the accuracy of your location of C by calculating the distances of A and B from C and comparing with the measurements on your drawing.

109. Locate on paper the four drill holes shown in Fig. 78.

110. Construct an angle equal to the angle in Fig. 79 and then bisect it

111. Find the length of the radius of the arc in Fig. 80.

112. Draw a line $4\frac{1}{2}$ in. long and divide it into five equal parts.

113. Lay out carefully on paper the holes for a jig as shown in Fig. 81.

114. In Fig. 82 compute the length of the line AB.

CHAPTER XI

CONSTRUCTION AND PROPERTIES OF GEOMETRIC FIGURES

61. Polygons.—A plane figure bounded by any number of straight lines can be given the general name of *Polygon.* There are special names for polygons according to the number of sides. A three-sided figure is called a *Triangle;* a four-sided figure, a *Quadrilateral;* a five-sided figure, a *Pentagon;* a six-sided figure, a *Hexagon;* an eight-sided figure, an *Octagon;* a ten-sided figure, a *Decagon.*

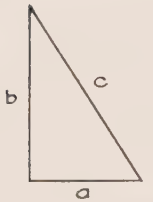
A *Regular Polygon* is a polygon having equal sides and equal angles.

The corners of a polygon are called its *Vertices.*

The sum of all the sides of a polygon is called the *Perimeter.*

62. The Triangle.—A Triangle is a polygon having but three sides. This is the smallest possible number of lines that could be used to enclose a figure. There are several special forms of triangles.

The *Right Triangle,* or right-angled triangle, has two of its sides meeting at right angles, as shown in Fig. 83. The side opposite



RIGHT TRIANGLE

Fig. 83.

ACUTE TRIANGLE

Fig. 84.

OBTUSE TRIANGLE

Fig. 85.

the right angle is called the *Hypotenuse.* The relation between its sides gives us one of the most useful laws of mathematics; namely, *the square of the hypotenuse is equal to the sum of the squares of the other two sides.* In Fig. 83, $c^2=a^2+b^2$. This relation is useful in calculating the lengths of diagonal distances and in laying out right angles by the use of three numbers having the relation,

such as 3, 4, and 5. This law is often called the *Pythagorean theorem*, from Pythagoras, the ancient Greek philosopher who discovered it. It must be remembered that this relation is true only for a right triangle.

If all of the angles of a triangle are acute angles, as in Fig. 84, the triangle may be called an *Acute Triangle.*

If one of the angles of a triangle is an obtuse angle, as in Fig. 85, the triangle may be called an *Obtuse Triangle.*

When a triangle has two of its sides equal, as shown in Fig. 86, the triangle is called an *Isosceles Triangle.* The angles opposite these sides are also equal; in Fig. 86 $\angle CAB = \angle CBA$.

When a triangle has all three sides equal, the triangle is called an *Equilateral Triangle.* The three angles are also equal. In Fig. 87, $AB = AC = BC$; also $\angle CAB = \angle CBA = \angle ACB$.

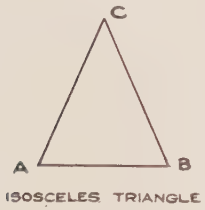


Fig. 86.

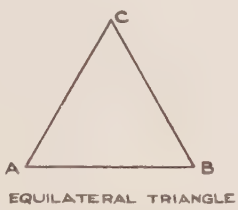


Fig. 87.

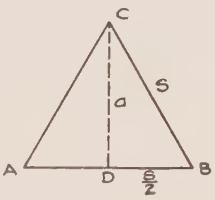


Fig. 88.

The altitude or height of an equilateral triangle is $\frac{\sqrt{3}}{2}$ times one side of the triangle. Let the triangle of Fig. 88 represent any equilateral triangle with each side s and the altitude a. The altitude will divide the triangle into two equal right triangles and the base into two equal parts, each $\frac{s}{2}$ in length. Then, considering the right triangle CDB, we will have

$$a^2 = s^2 - \left(\frac{s}{2}\right)^2 = s^2 - \frac{s^2}{4}$$

$$= \frac{4}{4}s^2 - \frac{s^2}{4} = \frac{3}{4}s^2$$

$$a = \sqrt{\frac{3}{4}s^2} = \frac{\sqrt{3}}{\sqrt{4}} \times \sqrt{s^2}$$

$$a = \frac{\sqrt{3}}{2}s$$

Since $\sqrt{3} = 1.732+$ this formula can be reduced to

$$a = .866\ s$$

If we wish to get the side when the altitude is given, we can transform this formula as follows:

$$a = .866\ s$$

$$s = \frac{a}{.866} = \frac{1}{.866}\ a$$

$$s = 1.155\ a$$

Put in words, this means that one side of an equilateral triangle equals 1.155 times the altitude.

If any triangle is cut out of paper, and then the three corners are torn off as shown in Fig. 89 and placed with their vertices

Fig. 89.

together, the three angles will form a straight line or an angle of 180°. In other words, the sum of the three angles of any triangle equals two right angles or 180°. From this it will be seen that each angle of an equilateral triangle equals $\frac{1}{3}$ of $180° = 60°$.

63. The Square.—The best known quadrilateral or four-sided figure is the *Square*, which has all four sides equal and all of the angles right angles (Figs. 90 and 91).

The method of constructing a square on a given line involves the erecting of perpendiculars as explained in the last chapter. Having erected a perpendicular, AC, Fig. 90, at one end of the given line, AB, and having marked it off at the right length, the remaining corner D can be located by striking intersecting arcs from C and B as centers with a radius equal to AB.

To inscribe a square in a circle, draw two perpendicular diameters. Their extremities on the circle will be the corners of the square.

It is frequently of advantage to know the relation between one side of a square and the diagonal of the square. (The Diagonal

is the line connecting the opposite corners.) This relation is useful in calculating the space needed in which to turn a square nut. It is also used in milling a square on a round bar, since the diameter of the bar is the diagonal of the largest square that can be

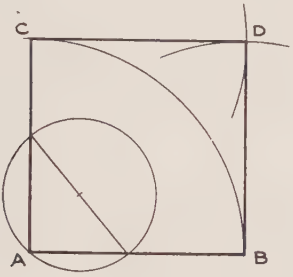


FIG. 90.

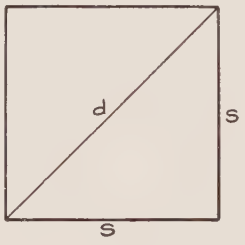


FIG. 91.

milled from the bar. In Fig. 91 it will be seen that two sides of a square with the diagonal form a right triangle.

Hence,

$$d^2 = s^2 + s^2 = 2\ s^2$$

$$d = \sqrt{2\ s^2} = \sqrt{2} \times \sqrt{s^2}$$

$$d = \sqrt{2} \times s = 1.414\ s$$

Then the diagonal of any square equals $\sqrt{2}$, or 1.414, times the length of one side of the square.

When the length of one side is wanted in terms of the diagonal, we can get

$$2\ s^2 = \quad d^2$$

$$s^2 = .5\ d^2$$

$$s\ = \sqrt{.5} \times d = .707\ d$$

This can also be obtained by transforming the formula $d = 1.414\ s$.

$$s = \frac{d}{1.414} = \frac{1}{1.414}\ d = .707\ d$$

Example:

What space is needed to turn a square nut for a $\frac{3}{4}$-in. bolt?

The width of nut $= \frac{1}{2}D + \frac{1}{8}$ in.

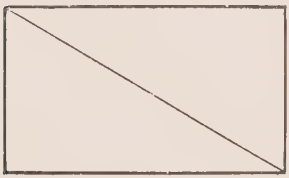
$= 1\frac{1}{2} \times \frac{3}{4} + \frac{1}{8} = 1\frac{1}{4}$ in.

The diagonal of the nut $= 1.414 \times 1\frac{1}{4} = 1.77$ in.

The nut must therefore have a clear circle 1.77 in. in diameter in which to turn. In most cases it is necessary to add to this an allowance for the jaws of the wrench.

64. The Rectangle.—The Rectangle is similar to the square in that its opposite sides are parallel and its angles are all right angles but, as shown in Fig. 92, the sides of the rectangle are not necessarily all equal. If the rectangle is divided into two parts by a diagonal, it will be seen that two equal right triangles are formed.

65. The Parallelogram.—The Parallelogram is a quadrilateral, the opposite sides of which are equal and parallel, as shown in

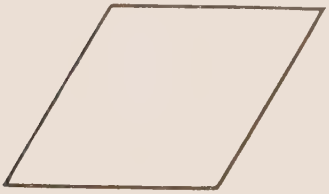


RECTANGLE
Fig. 92.

PARALLELOGRAM
Fig. 93.

Fig. 93. The diagonally opposite angles in a parallelogram are equal. Any two adjacent angles are supplementary angles, their sum being 180°. It will be noticed that the rectangle is a special case of a parallelogram, or a rectangle is a parallelogram having right angles.

When a parallelogram has all four sides equal as in Fig. 94,



RHOMBUS
Fig. 94.

TRAPEZOID
Fig. 95.

thus resembling a square except that it does not have right angles at the corners, it is called a *Rhombus*.

66. The Trapezoid.—The Trapezoid is a quadrilateral with only two sides parallel. (See Fig. 95.) Pieces of this shape are often encountered in sheet metal work and in boiler and tank work. Railroad embankments and cuts have trapezoidal sections, as do also some dams.

67. The Pentagon.—A Pentagon is any figure having five sides, but we generally think of it as meaning a *regular* pentagon; that is, one having equal sides and equal angles. The regular pentagon is not often seen in a shop. One familiar use, however, is on the valve stems of fire hydrants, which have pentagonal heads so that they can be opened only with a special wrench. There may be times when one wishes to construct a star-shaped figure or to lay out five holes equally spaced on a circle, so it is well to know how to construct a regular pentagon.

To inscribe a pentagon in a circle. Having the circle as in Fig. 96, draw two diameters AB and CD at right angles to each other. Bisect the radius CO, thus locating its middle point E. From E as center and with the distance EA as radius, swing an arc from A cutting the diameter CD at G. The distance AG is equal to the side of the pentagon. Starting from any point,

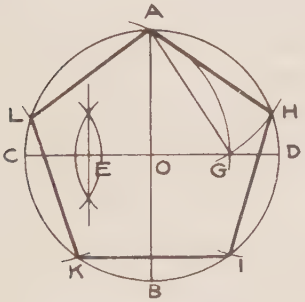


Fig. 96.

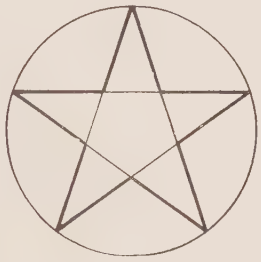


Fig. 97.

such as A, draw AH, then HI, IK, KL, and LA each equal to AG. Then $AHIKL$ is the desired pentagon.

Note.—A five-pointed star may be constructed within a circle by dividing the circle, in the method just described, into five equal parts and then connecting each point to the second point on each side of it by straight lines (see Fig. 97).

To construct a pentagon having a given length of side. Let OB, Fig. 98 be the given length of side. With O as center and OB as radius describe a semicircle and extend BO to A. With a protractor, or by trial with the dividers, divide this semicircle into five equal parts, giving the points of division C, D, E, and F. From D, the second point from A, draw a line to O. From O, through E and F draw lines of indefinite length. From D with OB as radius scribe an arc cutting the line through E at G.

Likewise from B scribe an arc cutting the line through F at H. Draw DG, BH, and GH, thus completing the pentagon.

Note.—The angle between two adjacent sides of any regular polygon of n sides is $\frac{n-2}{n} \times 180°$. Hence the method just given may be used to construct a regular polygon of any number of sides by dividing the semicircle into as many parts as there are to be sides in the polygon and discarding the first two divisions from A. For drawing the square, hexagon, and octagon, the special constructions given are more convenient.

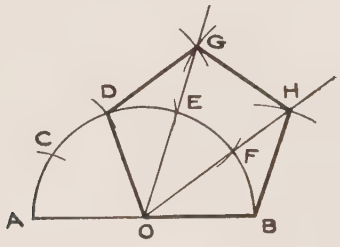


FIG. 98.

68. The Hexagon.—The regular hexagon or six-sided figure is frequently seen in the shop, especially in nuts and in bolt and cap-screw heads. The word is often abbreviated to "hex" in the shop. Thus a "hex" nut would mean a hexagonal nut. If the six corners of a regular hexagon are connected to the center, six equilateral triangles are formed, as shown in Fig. 99. From this it is evident that the circumscribing circle (the circle passing through the corners) has a radius equal to one of the sides of the hexagon. This is in accord with the statement in the preceding chapter that a chord equal to the radius of a circle would subtend (mark off) an arc of 60°. This readily indicates the methods of constructing a hexagon.

To inscribe a hexagon in a circle. Having the circle as in Fig. 99, set the dividers to the radius of the circle and step around the circle, thus dividing the circumference into six equal parts. Connect the points thus located, forming a regular hexagon within the circle.

To construct a hexagon having a given length of side. In the preceding construction the side of the hexagon is equal to the radius of the circumscribing circle. Hence, to construct a hexa-

gon having a given side, it is but necessary to draw a circle with the given side as radius and then to proceed as before and inscribe a hexagon in the circle.

69. Long and Short Diameters of Hexagons.—There are three well-known methods of designating the size of a hexagon. One way consists in giving the length of one side of the figure, as AB in Fig. 100. Another is to give the distance between the opposite corners, as AD in Fig. 100. This distance is popularly called the *long diameter* or the *distance across the corners*. The third and

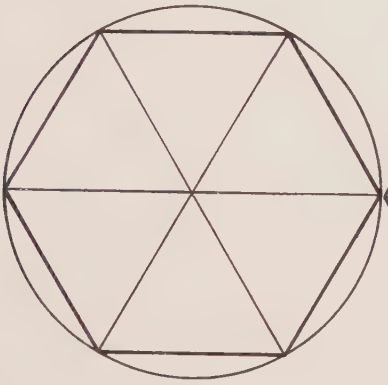


Fig. 99.

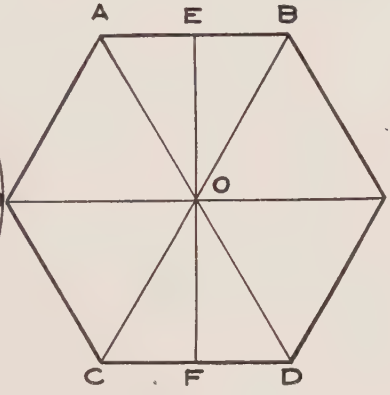


Fig. 100.

most common method of designation is to give the perpendicular distance between the opposite sides, as EF in Fig. 100. This distance is commonly known as the *short diameter* or the *distance across the flats*. This is the measurement by which hexagons are commonly designated in the shop. For example, a $1\frac{1}{2}$-in. hex bar would mean a hexagonal bar having the distance EF in Fig. 100 equal to $1\frac{1}{2}$ in. The use of this measurement has probably sprung from the fact that this distance is the width of opening for a wrench for a hexagonal nut or bolt-head. Although this measurement is used in designating hexagons, it is desirable, if it becomes necessary to lay out a hexagon, to have either the distance across corners or the length of one side. Consequently, a knowledge of the relations of these measurements is useful.

As explained previously, the hexagon consists of six equilateral triangles. Consequently, in Fig. 100, AOB is an equilateral triangle and the side $AB = AO = BO$. But BO is one-half of the

distance across corners BC. Hence the side AB is just one-half of the long diameter BC.

The point E is the middle point of AB; and EO is the altitude of the equilateral triangle AOB. As explained in studying the equilateral triangle, the altitude equals $\frac{1}{2}\sqrt{3}$, or .866, times the side.

$$EO = .866\ BO$$

But EO is one-half of the distance across flats, EF; and BO is one-half of the distance across corners, BC. Multiplying the above equation by 2 we get

$$2\ EO = .866 \times 2\ BO$$
$$EF = .866\ BC.$$

Distance across flats $= .866 \times$ *distance across corners.*

$$BC = \frac{1}{.866}\ EF = 1.155\ EF$$

Distance across corners $= 1.155 \times$ *distance across flats.*

These relations are useful in laying out hexagons of a given size (short diameter); in getting the size of round stock from which to cut a given hexagon; or in calculating the space needed in which to turn a hexagonal nut.

Example:

What distance from the bolt center must be allowed in order to be able to turn a "hex" nut for a 1-in. bolt?

The required distance from the bolt center to be left clear must be at least one-half the long diameter of the hexagon.

The short diameter of the nut is $1\frac{1}{2}\ D + \frac{1}{8}$ in. $= 1\frac{5}{8}$ in.

From the preceding discussion the long diameter of the hexagon will be

$$1.155 \times 1\tfrac{5}{8} = 1.877 \text{ in.}$$

The clearance from the bolt center must be at least half of this, or a little over $\frac{1}{10}$, so we would probably leave at least 1 in., and more if possible, to give freedom in handling the wrench.

70. The Octagon.—The regular octagon or eight-sided figure is most often seen in the shop in the stock for cold chisels and similar tools. The angle between two adjacent sides of an octagon is 135°, which is 90° + 45°, or a right angle plus one-half a right angle.

To inscribe an octagon in a circle. For this it is necessary first to draw two perpendicular diameters. These will locate every other corner of the octagon. To find the other corners we can bisect the right angles formed by these diameters.

To construct an octagon with a given side. The fact that the

corner angles are each $1\frac{1}{2}$ times a right angle gives a clue to the method used. Let AB, Fig. 101, be the given line. At A and B erect two perpendiculars AH and BI of indefinite length. With A and B as centers and AB as radius, describe semi-circles and extend AB to D and C. Bisect the angles MBD and LAC, and draw BF and AE. These will be sides of the octagon. Draw EG and FK parallel to AH and BI and equal to AB. With a radius equal to AB and with G and K as centers, strike arcs cutting the perpendiculars at H and I. Draw GH, KI, and HI.

To convert a square into an octagon. Sometimes we want to cut off the corners of a square to make an octagon. The proper

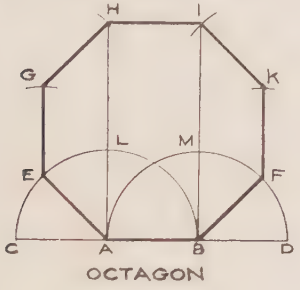


Fig. 101.

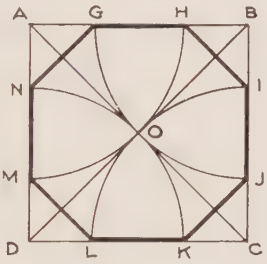


Fig. 102.

amount to be cut off can be determined by a simple geometrical construction.

Let $ABCD$, Fig. 102, be the given square. Draw the two diagonals AC and BD. With the corners A, B, C and D as centers and with $AO = BO = CO = DO$ as radius, strike the arcs HM, GJ, IL, and KN. Join the points at which these arcs cut the sides of the square, thus getting the other sides GN, HI, JK, and LM of the octagon.

71. Table for Spacing Circles.—For spacing a circle into any number of equal parts, it is necessary to know the length of the chord from one point to the next. In the cases of some of the regular polygons discussed in this chapter it is possible to arrive at the length of the chord of the circle by some geometric construction. The following table will enable a workman to divide a circle into any number of parts up to 100. The table gives the length of chord or the setting of the dividers to space a circle of 1-in. diameter into any desired number of equal parts. For circles of other diameters the length of the chord is obtained by multiplying the length given in the table by the diameter of the

circle. For example, to divide an 18-in. circle into 10 equal parts, the dividers should be set $18 \times .3090 = 5.562$ in.

LENGTHS OF CHORDS FOR SPACING A CIRCLE WHOSE DIAMETER IS 1

For Circles of Other Diameters, Multiply Length of Chord given in Table by Diameter of Circle

No. of spaces	Length of chord	No. of spaces	Length of chord	No. of spaces	Length of chord	No. of spaces	Length of chord
.....	.....	26	.1205	51	.0616	76	.0413
.....	.....	27	.1161	52	.0604	77	.0408
3	.8660	28	.1120	53	.0592	78	.0403
4	.7071	29	.1081	54	.0581	79	.0398
5	.5878	30	.1045	55	.0571	80	.0393
6	.5000	31	.1012	56	.0561	81	.0388
7	.4339	32	.0980	57	.0551	82	.0383
8	.3827	33	.0951	58	.0541	83	.0378
9	.3420	34	.0923	59	.0532	84	.0374
10	.3090	35	.0896	60	.0523	85	.0370
11	.2817	36	.0872	61	.0515	86	.0365
12	.2588	37	.0848	62	.0507	87	.0361
13	.2393	38	.0826	63	.0499	88	.0357
14	.2225	39	.0805	64	.0491	89	.0353
15	.2079	40	.0785	65	.0483	90	.0349
16	.1951	41	.0765	66	.0476	91	.0345
17	.1838	42	.0747	67	.0469	92	.0341
18	.1736	43	.0730	68	.0462	93	.0338
19	.1646	44	.0713	69	.0455	94	.0334
20	.1564	45	.0698	70	.0449	95	.0331
21	.1490	46	.0682	71	.0442	96	.0327
22	.1423	47	.0668	72	.0436	97	.0324
23	.1362	48	.0654	73	.0430	98	.0321
24	.1305	49	.0641	74	.0424	99	.0317
25	.1253	50	.0628	75	.0419	100	.0314

72. The Ellipse.—The Ellipse, Fig. 103, is a curved figure which may be defined very briefly as a flattened circle. It is used for manholes and handholes in boilers. Curved arches are generally made to the shape of half an ellipse. When an ellipse is to be drawn, the long and short diameters AB and CD, Fig. 103, are usually given.

To draw an ellipse (*exact method*). Draw the two diameters

AB and *CD*, Fig. 103, at right angles to each other. With *AO* as radius and *C* as center, scribe an arc cutting *AB* at two points *E* and *F*. These points are called the *focii* (plural of *focus*) of the ellipse. Fix two pins at *E* and *F* and loop a thread or cord around *E* and *F* so that when pulled taut it will just reach *C*.

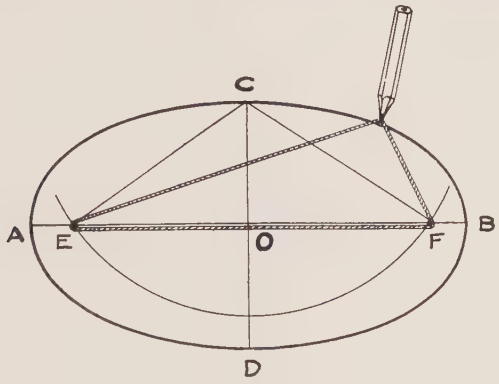


Fig. 103.

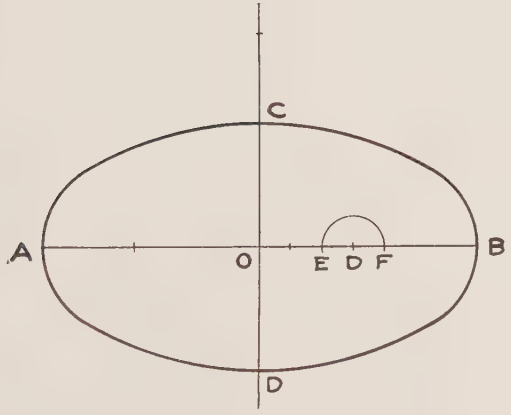


Fig. 104.

Place a pencil inside the cord and, keeping the cord in even tension, move the pencil around, thus describing the ellipse.

To draw an ellipse (approximate method). There have been many methods devised by draftsmen to draw a curve with the compasses that will be very near an ellipse in appearance. One of the simplest of these is as follows: Draw the long and short diameters *AB* and *CD* in Fig. 104. From *B* set in toward *O* a distance *BD* equal to *CO*, leaving *OD* equal to the difference

between OB and OC. With the eye, divide OD into three equal parts and swing one of the parts DE, back towards B, giving the point F. The long diameter is thus divided into two lengths AF and BF. The length BF can be used as a radius for drawing the ends of the ellipse and AF can be used as a radius for drawing the flat sides.

PROBLEMS

115. Construct a triangle having sides 2 in., $3\frac{1}{2}$ in., and $2\frac{1}{2}$ in. in length. What kind of a triangle is this?

116. Using the method explained in Art. 55, Chapter X, copy the angles of this triangle, placing the angles together as shown in Fig. 89, and thus prove that the sum of the three angles is 180°.

117. A roof truss, Fig. 105, has the top "chords" AB and BC each 27 ft. 1 in. long while the bottom chord AC is 45 ft. long. What is the height of the truss in the center?

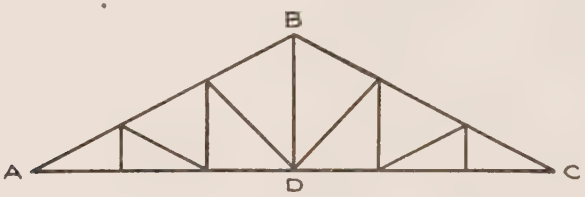


FIG. 105.

118. A round shaft 2 in. in diameter is to be milled square on one end. What is the length of side of the largest square that could be cut from the shaft?

119. (a) Calculate the size of circle required in which to turn a finished square nut for a $1\frac{1}{4}$-in. bolt. (Width of finished nut $= 1\frac{1}{2}\,D + \frac{1}{16}$ in.)

(b) Calculate the same for a finished "hex" nut.

120. Construct a five-pointed star in a 2-in. circle.

121. The sides of the heads of all standard hexagon-head cap-screws $\frac{1}{2}$-in. or over in diameter project $\frac{1}{8}$ in. beyond the body of the bolt. In other words, the width of the head $= D + \frac{1}{4}$ in., where D is the diameter of the screw. What size of hexagon-head cap-screws could be cut from a $1\frac{1}{8}$-in. hex bar?

122. The corners of a 3 in. square bar are to be cut off to make an octagon. Calculate the length of each face of the octagon.

Hint: Referring to Fig. 102, calculate AC, then OC $(=IC)$, then BI, then HI.

123. Construct an approximate ellipse having a long diameter of 4 in. and a short diameter of $2\frac{1}{2}$ in.

124. Construct a pentagon, a hexagon, and an octagon each 1 in. on a side.

CHAPTER XII

AREAS OF GEOMETRIC FIGURES

73. Area of a Square or Rectangle.—The method of calculating the areas of rectangular figures is so well known as to need little if any explanation at this time. The most common unit of area is the *square inch*, although square feet, square yards, square rods, and square miles are sometimes used for areas involving larger dimensions.

The square inch is the amount of surface in a square 1 in. on each side, see Fig. 106. Likewise a square foot is the area of a square 1 ft. on each side, and similarly for the other units. The number of square inches in a square other than 1 in. on each side

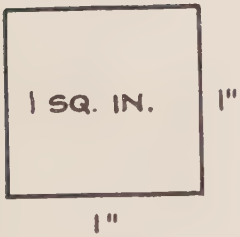


Fig. 106.

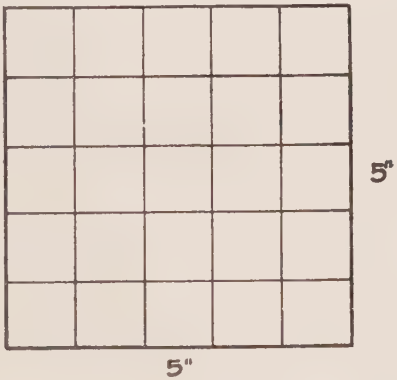


Fig. 107.

is the product of the two sides or, in other words, it is the *square* of one side. In Fig. 107, if each side of the square is 5 in., the figure will contain 5^2 or 25 sq. in.

In a rectangle, the area is the product of the two dimensions. If Fig. 108 represents a sheet of boiler plate 5 ft. by 2 ft., the area of the plate is 5×2 or 10 sq. ft.

74. Area of a Triangle.—Any triangle is readily seen to have just one-half the area of a rectangle having the same base and altitude. In Fig. 109, the triangle ABE = triangle ABD.

Likewise triangle DBC = triangle CBF. Thus, the triangle ABC is shown to have just half the area of the rectangle $AEFC$. But the area of the rectangle is the base AC times the altitude DB. Hence, the area of the triangle $ABC = \frac{1}{2}$ of $AC \times BD$.

In a right triangle, one side can be taken as the base and the other side as the altitude. Then the area of a right triangle is one-half the product of the two sides forming the right angle.

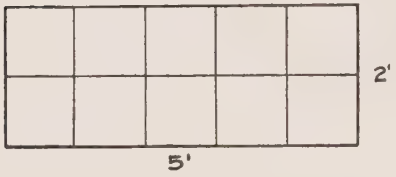


Fig. 108.

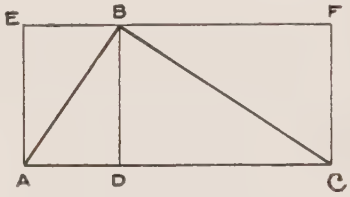


Fig. 109.

In an equilateral triangle we found (Art. 62) that the altitude is .866 s (see Fig. 88). If the base is s and the altitude .866 s, the area is

$$A = \tfrac{1}{2} \text{ of } .866\ s \times s$$
$$= .433\ s^2$$

In case the three sides of a triangle are known but not the altitude, the area can be found as follows:

(1) Add together the three sides and divide by 2.

(2) From this half sum, subtract each side separately.

(3) Multiply together the half sum and these three remainders and extract the square root of the product. The result is the area. As a formula this is indicated by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the three sides of a triangle and $s = \dfrac{a+b+c}{2}$

Example:

A triangular plate has sides 12, 10, and 6 in. long. What is its area?

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$s = \frac{12+10+6}{2} = 14$$
$$A = \sqrt{14\ (14-12)(14-10)(14-6)}$$
$$= \sqrt{14 \times 2 \times 4 \times 8}$$
$$= \sqrt{896} = 29.93 + \text{sq. in., } \textit{Answer.}$$

75. Area of a Parallelogram.—In the parallelogram of Fig. 110, if the triangle DEC is removed and placed at the other end, as

marked at *AFB*, the parallelogram is converted into a rectangle *AFED* without changing the area. The area of the rectangle is *AD* times *DE*. But *DE* is the altitude of the parallelogram, or the perpendicular distance between the two bases. Therefore, the area of a parallelogram is the product of the base times the altitude. Notice particularly that the length of the base is multiplied by the *perpendicular* distance between the two faces.

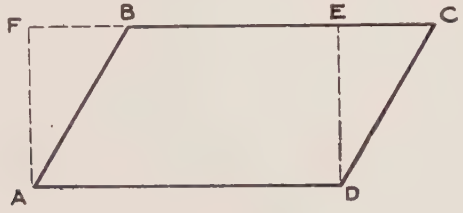


Fig. 110.

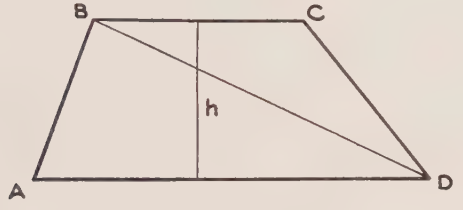


Fig. 111.

76. Area of a Trapezoid.—If, in the trapezoid *ABCD* of Fig. 111, we draw the diagonal *BD*, it will cut the trapezoid into two triangles *DBC* and *ABD*. The triangle *DBC* has a base *BC* and an altitude *h*. The triangle *ABD* has a base *AD* and the same altitude *h*.

$$\text{Area of triangle } BDC = \tfrac{1}{2}\, BC \times h$$
$$\text{Area of triangle } ABD = \tfrac{1}{2}\, AD \times h$$
$$\text{Area of trapezoid } ABCD = \tfrac{1}{2}\, BC \times h + \tfrac{1}{2}\, AD \times h$$
$$= (\tfrac{1}{2}\, BC + \tfrac{1}{2}\, AD)h$$
$$= \tfrac{1}{2}(BC + AD)h$$

If we let b and b' stand for the two bases, we get the formula

$$\text{Area of a trapezoid} = \frac{(b+b')h}{2}$$

Example:

A sheet of $\frac{3}{8}$-in. steel plate 36 in. wide is cut so that it is 12 ft. long on one edge and $8\frac{1}{2}$ ft. long on the other edge. What is the weight of the plate?

$$A=\frac{(b+b')h}{2}$$
$$=\frac{(12+8.5)3}{2}$$
$=30.75$ sq. ft.
$5\times1.02=5.1$
$3\times5.1\ =15.3$
$30.75\times15.3=470.5$ lb., *Answer.*

Explanation: The plate forms a trapezoid having bases 12 ft. and 8.5 ft. long and an altitude of 3 ft. Its area is therefore 30.75 sq. ft. In Vol. I, Chapter X, it was stated that a wrought iron plate 1 ft. square and $\frac{1}{8}$ in. thick weighs 5 lb. and that steel is 2 per cent. heavier. Hence 1 sq. ft. of steel plate $\frac{1}{8}$ in. thick weighs 5.1 lb. and 1 sq. ft. $\frac{3}{8}$ in. thick weighs 3×5.1, or 15.3 lb. The whole plate weighs 30.75×15.3, or about 470.5 lb.

77. Area of a Trapezium.—Trapezium is a name sometimes used for an irregular quadrilateral that does not have any two sides parallel (see Fig. 112). To get the area of such a figure, the best plan is to draw a diagonal such as AC, Fig. 112, thus dividing the figure into two triangles. The only measurements necessary are the diagonal AC and the two perpendiculars upon it, BE and FD. AC is the base of both triangles and their altitudes are BE and FD.

$$\text{Area of } ABCD=\frac{AC\times BE}{2}+\frac{AC\times FD}{2}$$
$$=\frac{AC(BE+FD)}{2}$$

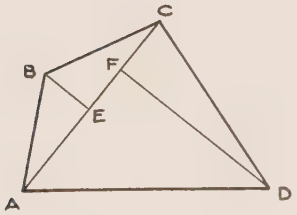


Fig. 112.

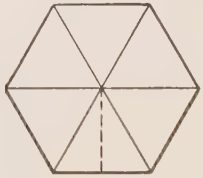


Fig. 113.

78. Area of a Hexagon.—The area of a hexagon is figured as follows (see Fig. 113). The area of one of the six equilateral triangles equals $\frac{1}{2}$ the base times the altitude, or

$\frac{1}{2}\times$ side $\times(.866\times$ side$)=\frac{1}{2}\times.866$ (side)$^2=.433$ (side)^2

For six triangles or the whole hexagon we would have:

Area of hexagon $=6\times.433$ (side)^2
$\therefore$ Area of hexagon $=2.598$ (side)^2

This is one way of calculating the area but, as will be seen, it is based on the length of one side of the hexagon. For convenience in shop work this can be changed into a formula involving only

the short diameter, by which the hexagon is best known. We have shown that the long diameter is 1.155 times the short diameter. Since the length of one side is $\frac{1}{2}$ of the long diameter, we have:

Side $= \frac{1}{2}$ long diameter $= \frac{1}{2}$ of $1.155 \times$ short diameter
$= .577$ short diameter
Area $= 2.598\ (\text{side})^2 = 2.598 \times (.577 \times \text{short diameter})^2$
$= 2.598 \times .577^2 \times (\text{short diameter})^2$
$= .866 \times (\text{short diameter})^2$

79. Area of any Regular Polygon.—Constants corresponding to 2.598 can be worked out for calculating the area of other regular polygons from the length of one side. In the case of the square, the area = side^2, so the constant equals 1.

CONSTANTS FOR CALCULATING THE AREAS OF REGULAR POLYGONS

Number of sides	Constant	Number of sides	Constant
3	.433	8	4.828
4	1.	9	6.182
5	1.721	10	7.694
6	2.598	11	9.366
7	3.634	12	11.196

To get the area of any regular polygon. Find the length of one side, square it, and multiply this by the constant given in the table. As a formula this would be

$$A = CS^2$$

80. Area of an Irregular Polygon.—To get the area of any irregular figure bounded by straight lines, divide the figure into triangles and rectangles or squares of which the areas can be readily calculated. Fig. 114 shows a sheet-metal bracket, the area of which can be calculated by dividing the figure into two rectangles and two triangles.

81. Area of a Circle.—The area of a circle equals $\frac{\pi}{4}$, or, .7854, times the square of the diameter. This should need little ex-

planation at this time, but the following simple demonstration may be of interest.

Imagine a circle to be divided into a number of triangles meeting at the center as in Fig. 115. If these triangles are then strung out as shown in the figure, it will be seen that the altitude of each of the triangles is the radius of the circle, while the bases of all the

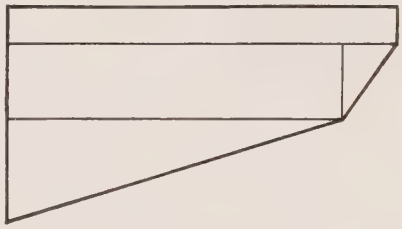


Fig. 114.

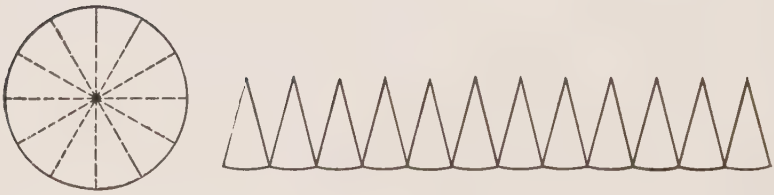


Fig. 115.

triangles together make up the circumference of the circle. The area of all the triangles equals one-half the circumference of the circle times the radius, or

$$A = \tfrac{1}{2}(2\pi\ r) \times r = \pi\ r^2$$

If we substitute $\frac{d}{2}$ for r we will get

$$A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi\ d^2}{4}$$

Hence, the area of a circle equals π times the square of the radius, or it equals $\frac{\pi}{4}$ times the square of the diameter.

Instead of using the expression $\frac{\pi}{4}$, its numerical value .7854 is often used, giving the formula $A = .7854\ d^2$. It is therefore evident that a circle has an area equal to .7854 of that of the circumscribing square.

82. Area of an Ellipse.—An ellipse being, in short, a flattened circle, its area is calculated in much the same way. The area of an ellipse equals $\frac{\pi}{4}$ times the product of the long and short diame-

ters, or π times the product of the long and short radii. In Fig. 116 if a and b represent the long and short diameters of the ellipse, we will have for the area:

$$A = \frac{\pi}{4} ab.$$

From this it will be seen that the ellipse bears the same relation to the circumscribing rectangle that the circle does to the circumscribing square.

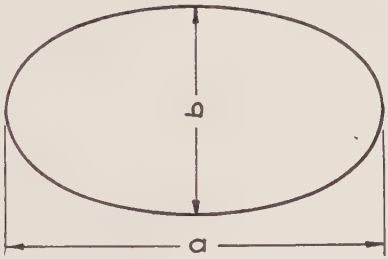


Fig. 116.

83. Area of a Sector of a Circle.—In calculating the area of a sector (Fig. 117) we determine it as a fraction of the complete circle. The fraction of the circle is the relation of the angle of the sector as compared with the 360° in the complete circle. A

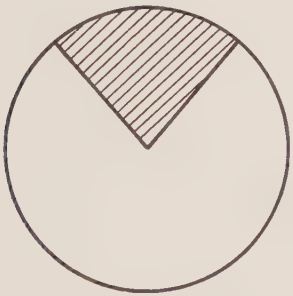


Fig. 117.

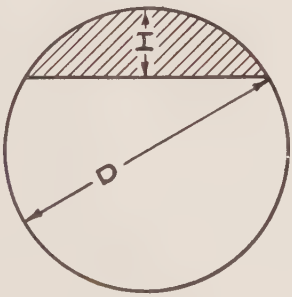


Fig. 118.

sector of 90° would be $90 \div 360 =$ one-fourth of a circle. Similarly a sector of 45° would be one-eighth of a circle.

If the radius and the length of the arc of a sector are known, the area can be calculated by taking one-half the product of the radius times the length of arc. To see this it is but necessary to imagine the sector to be made up of a number of small triangles having the

radius for a common altitude and having their bases together equal to the arc of the sector.

84. Area of a Segment of a Circle.—There are many rules for finding the area of a segment of a circle, depending on the dimensions given. In most cases the dimensions given are the diameter, D, of the circle and the height, H, of the segment, as shown in Fig. 118. In finding the amount of steam space or water space in a boiler, or in calculating the amount of oil in a horizontal cylindrical storage tank, we would have the diameter of the boiler or tank and the height of the segment from which to calculate the area of the segment. The following simple formula has been worked out for the area of a segment:

$$A = \frac{4}{3} H^2 \sqrt{\frac{D}{H} - .608}$$

This formula is not absolutely exact but is sufficiently accurate for all practical purposes.

Example:

A horizontal cylindrical tank 6 ft. in diameter and 24 ft. long is used for storing fuel oil. How many gallons are there in the tank when the oil is 2 ft. deep?

$$A = \frac{4}{3} H^2 \sqrt{\frac{D}{H} - .608}$$

$$= \frac{4}{3} \times 2^2 \sqrt{\frac{6}{2} - .608}$$

$$= \frac{4}{3} \times 4 \sqrt{2.392}$$

$$= \frac{16}{3} \times 1.546$$

$$= 8.245 \text{ sq. ft.}$$

$$V = 8.245 \times 24$$

$$= 197.88 \text{ cu. ft.}$$

$$197.88 \times \frac{1728}{231} = 1480\tfrac{1}{4} \text{ gal., } \textit{Answer.}$$

Explanation: The oil occupies a segment having a height $H = 2$ ft. in a circle of diameter, $D = 6$ ft. Substituting these values for D and H in the formula, we find the area of the segment to be 8.245 sq. ft. Since the tank is 24 ft. long, the volume of the oil is 197.88 cu. ft. There are 1728 cu. in. in 1 cu. ft., and a gallon contains 231 cu. in., so we multiply the number of cu. ft. by 1728 and divide by 231. This gives $1480\frac{1}{4}$ as the number of gallons of oil in the tank.

When the length of the chord and the height of a segment are given, the area can be calculated from the preceding formula after first calculating the diameter of the circle from the chord and the height.

$$D = \frac{C^2}{4H} + H$$

where D = the diameter, C = the chord, and H = the height of the segment.

85. Area of any Irregular Figure.—When a figure cannot be divided into regular shapes, some other means must be used to ascertain its area.

One simple method is to cut out the figure in paper or sheet metal and compare the weight of this figure with the weight of 1 sq. in. cut from the same material.

The following approximate method of calculating areas is often used to get the areas of indicator cards from steam or gas engines but can be applied to any irregular figure: Fig. 119 shows the method as applied to an indicator diagram from a steam engine. First draw any line such as AB, outside of the figure, for a reference line. Erect two perpendiculars so as to just touch the extremities of the figure. Divide AB between these perpendiculars into any number of equal parts by the method explained in Chapter X. At the *middle* of each of these divisions on AB erect perpendiculars crossing the figure and measure that part of each of these perpendiculars included between the upper and lower lines of the figure; add together, and divide the sum by the number of these lines. This will give the *average height* of the figure. This result multiplied by the length of AB will give the area. With ten or more divisions of AB the result thus found will seldom be off more than one or two per cent. The more divisions are used, the more accurate will be the result.

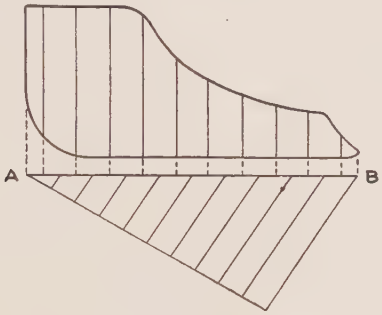


Fig. 119.

86. The Planimeter.—This is a mechanical device often used by draftsmen and engineers for measuring the areas of irregular figures. As shown in Fig. 120, it consists of two arms, A and B, hinged at C. There are pin points at D and E. That at D is fixed, while that at E is caused to trace around the outline of the figure to be measured. As E traces the outline of the figure, the

wheel at F rolls on the surface of the table. When E has completed the circuit, the graduations on F will indicate the area of the figure.

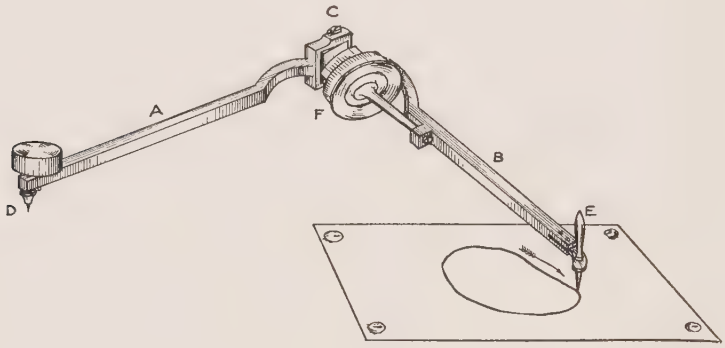


FIG. 120.

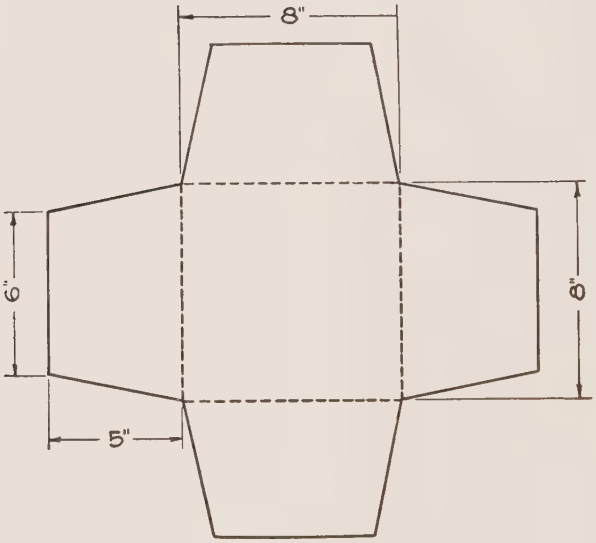


FIG. 121.

PROBLEMS

125. A firm uses on their product a brass name plate in the form of an equilateral triangle, $3\frac{1}{2}$ in. on each side. If these are pressed out of sheet metal of No. 12 B. & S. gage (= .0808 in. thick), what would be the weight of 10,000 of these plates? (Weight of brass per cu. in. = .301 lb.)

126. Find the dimensions of a square bar to weigh the same per foot as a $\frac{3}{4}$-in. round bar.

127. A 6 in. round pipe is to be replaced by an elliptical pipe, the short diameter of which is 4 in.; what must be the long diameter of the elliptical pipe to have the same area as the 6 in. pipe?

128. How many square feet of floor space in a four-story factory building of triangular shape, if the walls are 65, 75, and 80 ft. long?

129. Find the area of a piece of tin plate cut to the shape and dimensions shown in Fig. 121.

130. Find the sectional area of the moulding shown in Fig. 122.

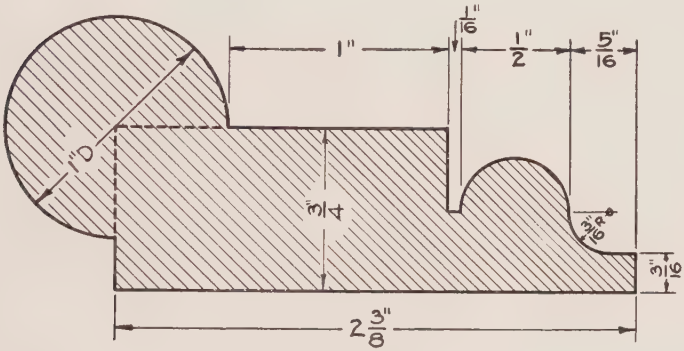


FIG. 122.

131. If a $\frac{5}{8} \times 2\frac{1}{4}$ in. hex cap-screw is turned from a $\frac{7}{8}$ in. hex bar, calculate the weight of metal removed in cutting. (Weight of steel per cubic inch = .283 lb.)

132. If four circular plates 24 in. in diameter are cut from a piece of boiler plate 48×48, calculate the per cent. of the stock that is wasted.

133. Find the area of Fig. 119 by the method described in Art. 85.

134. A horizontal cylinder boiler is 60 in. in diameter and 16 ft. long. If it is filled with water to a depth of 40 in., what is the volume of the steam space left above the water line?

CHAPTER XIII

VOLUMES AND SURFACES OF SOLIDS

87. The Prism.—The study of solids is of use to shop men chiefly in teaching how to calculate the quantity of material in a piece or the amount of sheet metal necessary to form the surface of an object.

In calculating volumes we usually base our calculations on the cubic inch as the unit of measure. The cubic foot is used occasionally for larger measurements. The contractor calculates excavations by the cubic yard.

The *Cubic Inch* is the volume of a cube (a block having six square faces) measuring 1 in. on each edge. See Fig. 123. Similarly, a *Cubic Foot* is the volume of a cube measuring 1 ft. on each edge; and a *Cubic Yard* is the volume of a cube measuring 1 yd. on each edge.

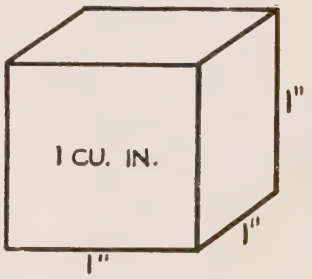


Fig. 123.

The simplest object of which to calculate the volume is the prism. A prism is a body having two opposite faces equal and parallel, the other faces being flat planes connecting the edges of the two parallel faces. See Fig. 124. The two equal and parallel faces are called the *bases* of the prism. The other faces are the *lateral* (side) *faces*. The lateral faces of a prism are all parallelograms and the lateral edges are therefore equal and parallel lines.

A *Right Prism* (Fig. 125) is one whose lateral edges are all perpendicular to the bases, so that the lateral faces are all rectangles.

A *Cube* is a prism bounded by six equal square faces. If we let e = one edge, then the area of one face = e^2. Hence, letting S = the total surface:

$$S = 6\ e^2$$

A hollow cube would be formed of sheet metal by cutting a sheet

containing six squares as in Fig. 126 and then bending it at right angles along the dotted lines.

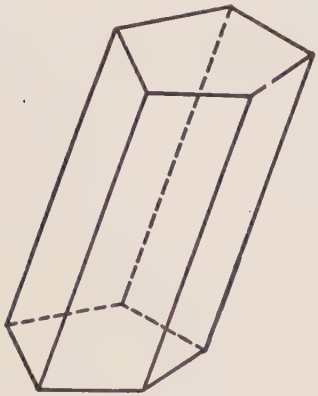


FIG. 124.—PRISM.

FIG. 125.—RIGHT PRISM.

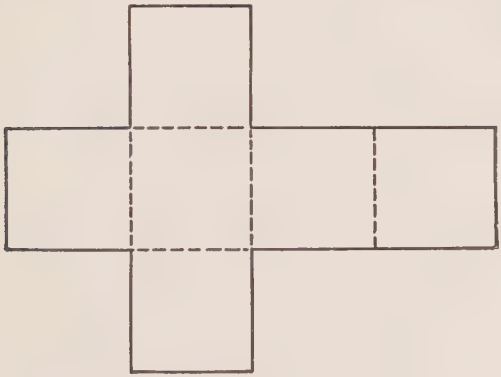


FIG. 126.

The volume of a cube is given by the formula:

$$V = e^3$$

The above formula is proved as follows:

Take a cube where $e = 5$ (Fig. 127). First consider the lowest slice. As shown in the lower figure this slice is 1 unit thick, 5 units wide, and 5 units long, and contains 5×5 or 25 cubic units. There are 5 such slices; hence the total volume equals $5 \times 25 = 125$.

$$V = 5 \times 5 \times 5 = 5^3 = 125$$

or, in general

$$V = e^3.$$

A *Rectangular Solid*, or *Rectangular Prism*, is a prism all of whose faces are rectangles. A brick is a rectangular prism. A cube is a special case of rectangular prism in which all the faces are squares.

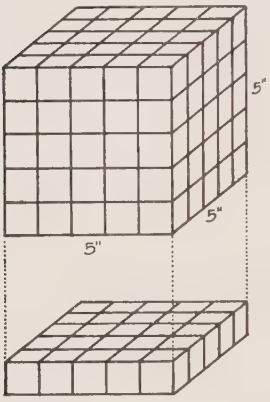


Fig. 127.

If we let a, b, and c, Fig. 128, represent the three dimensions of any rectangular solid, there will be two faces each $a \times b$, two faces $a \times c$, and two faces $b \times c$. Then the total surface of the rectangle will be

$$S = 2\,ab + 2\,ac + 2\,bc$$

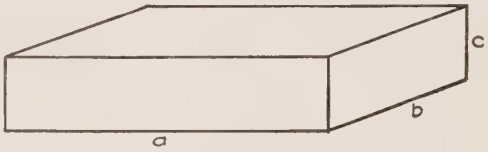


Fig. 128.

The volume or the number of cubic units is the product of the three dimensions, or

$$V = abc$$

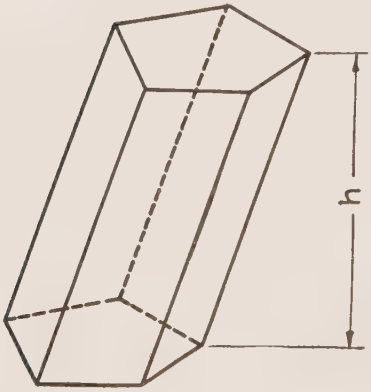


Fig. 129.

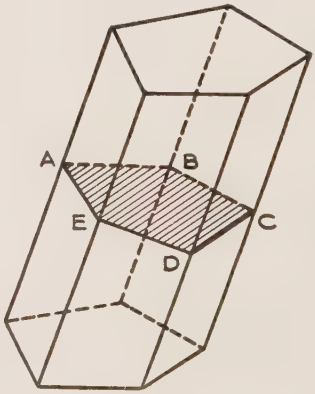


Fig. 130.

In any right prism the volume is obtained by multiplying the area of one base by the altitude, or the length of one of the lateral edges.

An *Oblique Prism* is a prism whose lateral edges are not per-

pendicular to the bases. Figs. 124 and 129 show oblique prisms.

The volume of an oblique prism may be calculated in either of two ways.

We may calculate the area of a base and multiply it by the altitude h, Fig. 129. (The altitude is the perpendicular distance between the bases.)

If more convenient, we may calculate the area of a section taken perpendicular to the edges ($ABCDE$, Fig. 130) and multiply it by the length of one edge. If a piece of regular stock is cut off diagonally on both ends so as to form an oblique prism, it would be simpler to follow the latter method, since the right section would be the regular section of the stock and the length of one edge would be easier to determine than the altitude.

88. The Cylinder.—A *Cylinder* is much like a prism, having parallel bases, but having a curved lateral surface. In geometry, the section of a cylinder may be of any curved shape; but in practical work we always think of a cylinder as being round, that is, of circular section. Any piece of straight, round, bar stock is a cylinder.

If a cylinder is cut off square at the ends as shown in Fig. 131, it is called a *Right Cylinder* and its bases are circles. Practically all the cylinders with which we have to do in the shop are right cylinders. Among these may be mentioned shafts, arbors, pins, pipes, and round rods of any sort. Steam engine cylinders, steam and water pipes, etc., are hollow cylinders, and the volume of metal contained in them is the difference between the volumes of the outer and inner cylinders.

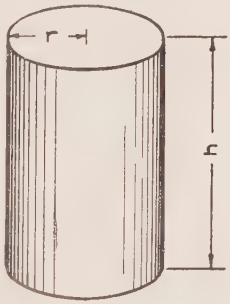


Fig. 131.

The volume of a solid right cylinder is obtained by multiplying the area of one end by the length of the cylinder

$$V = \pi\, r^2 h, \text{ or } .7854\, d^2 h$$

The lateral surface of a right cylinder may be considered as a rectangle wound about the cylinder; the height (h) of the rectangle equals the h of the cylinder, while the length of the rectangle is the perimeter of the cylinder, $2\,\pi\, r$, or $\pi\, d$. The area of the lateral surface is, therefore, $2\,\pi\, rh$. The ends are each circles of radius r. The total surface of a cylinder, when laid out flat,

therefore consists of a rectangle and two circles, as shown in Fig. 132.

89. Flow Through Pipes.—The quantity of water, gas, or other material flowing through a pipe in a given time may be considered as being a cylinder of matter having a sectional area equal to the sectional area of the inside of the pipe and having a length equal to the distance which the material flows in the given time. As a formula this may be written:

$$Q = AV$$

Q standing for quantity, A for area, and V for velocity.

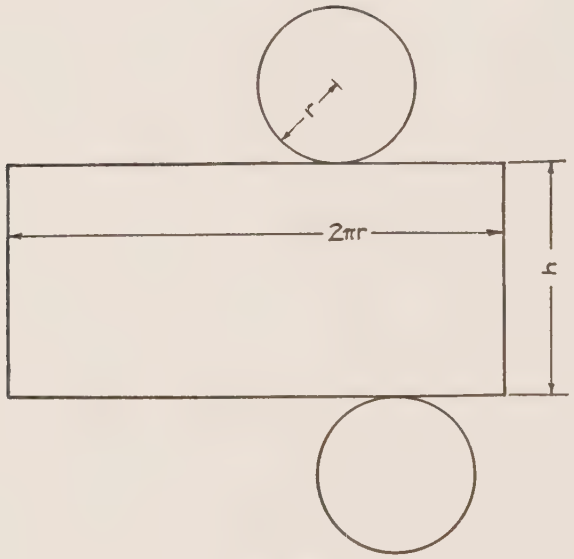


Fig. 132

Example:

How many gallons of water will be discharged per minute from a 4-in. pipe if it flows at a rate of 300 ft. per minute?

$Q = AV$
$= \pi \times 2^2 \times 3600$
$= 45{,}239$ cu. in.
$45{,}239 \div 231 = 195.8$ gal., *Answer.*

Explanation: The quantity per minute in cubic inches is the area in square inches times the velocity in inches per minute, or $Q = AV$. The velocity is $300 \times 12 = 3600$ in. per minute and the area is πr^2 or $\pi \times 2^2$. The discharge is therefore 45,329 cu. in. or 195.8 gal.

90. The Pyramid.—A *Pyramid* is a solid figure formed by a polygon, called the *base*, and a series of triangles meeting at a

common point called the *vertex* or *apex*. The pyramid shown in Fig. 133 has a parallelogram for a base.

A *Regular Pyramid* is one whose base is a regular polygon and whose vertex is directly above the center of the base. Fig. 134 shows a regular hexagonal pyramid.

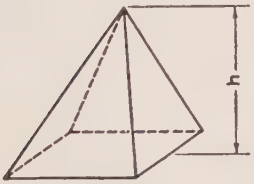


Fig. 133.—Pyramid.

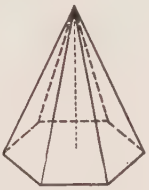


Fig. 134.

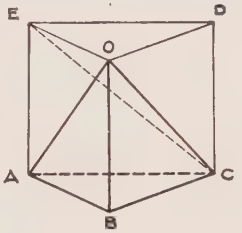


Fig. 135.

The volume of a pyramid is the product of the area of the base times one-third the altitude. In other words a pyramid contains one-third the volume of a prism having the same base and altitude. This is easily demonstrated from Fig. 135.

Let $O-ABC$ be a triangular pyramid and $ABC-EOD$ the triangular prism having the same base and altitude. The prism

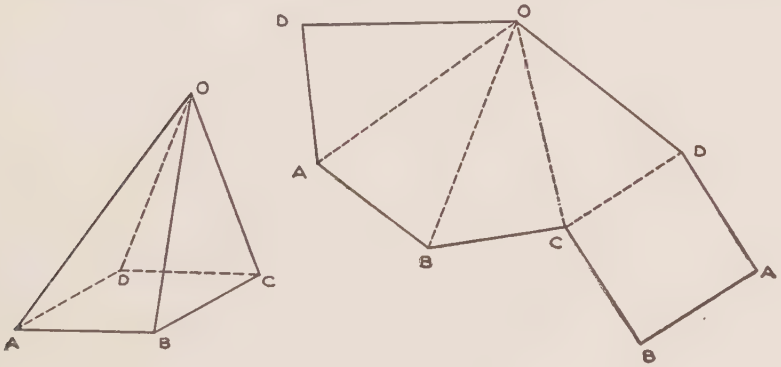


Fig. 136.

consists of three triangular pyramids $O-ABC$, $O-AEC$, and $O-EDC$.

Now $O-AEC$ and $O-EDC$ have the same altitude, and their bases AEC and EDC are equal triangles. Hence, these two pyramids are equal.

Again, pyramid $O-EDC$ may be regarded as having C for a

vertex and *EOD* for a base. Then this pyramid $C-EOD$ is equal to the pyramid $O-ABC$, because they have the same altitude and their bases *EOD* and *ABC* are equal.

Hence, the three pyramids are equal and each contains one-third the volume of the prism.

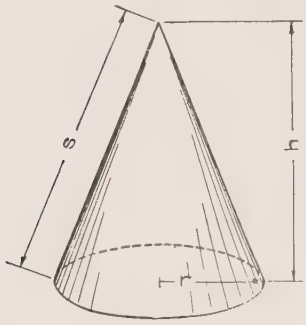


Fig. 137.

This proof can be extended to a pyramid having any number of sides in its base.

The surface of any pyramid can be laid out or developed on a flat surface and then folded into shape as shown in Fig. 136.

91. The Cone.—A *Cone* is much like a pyramid but has a circle for a base. The most common form is the *right cone* in which the apex is directly over the center of the base, as in the cone shown in Fig. 137. Like the pyramid, the volume of a cone is obtained by multiplying the area of the base by one-third the altitude.

$$V = \frac{\pi r^2 h}{3}$$

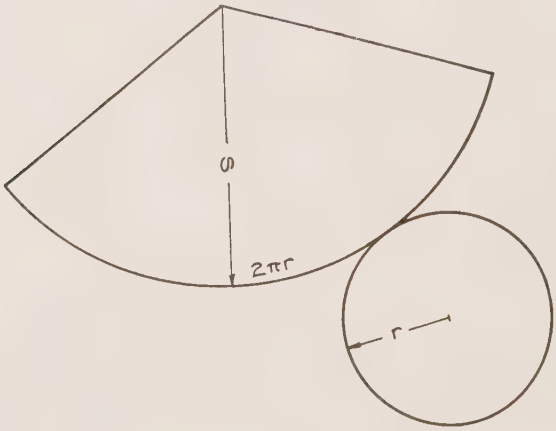


Fig. 138.

The surface of a cone can be quite readily laid out in sheet metal and it is frequently necessary to do this in a tin shop. In a right cone, every point in the edge of the base is equally distant from the apex. This distance (s in Figs. 137 and 138) is called the *slant*

height of the cone. The slanting surface of a cone is therefore a sector of a circle drawn with the slant height of the cone as a radius. The length of arc for the sector is obtained by laying off 2π times the radius of the base, or π or $\frac{22}{7}$ times the diameter, as in Fig. 138.

The slanting surface of a right cone may be considered as a triangle, with a base equal to the perimeter of the base of the cone and an altitude equal to the slant height. The total surface, including the base is

$$\begin{aligned} &\tfrac{1}{2} \times 2\pi r \times s + \pi r^2 \\ &= \pi rs + \pi r^2 \\ &= \pi r(s + r) \end{aligned}$$

92. Frustums of Cones and Pyramids.—A *Frustum* of a cone or pyramid is the part left after the top has been cut off by a plane parallel to the base. Fig. 139 shows a frustum of a pyramid. One way to get the volume of a frustum is to draw in the part cut off and calculate the difference between the volume of the whole cone and of the part cut off. By doing this, and then working out a formula which will contain only the dimensions ordinarily given, we arrive at the following rule for the volume of a frustum of either a pyramid or cone:

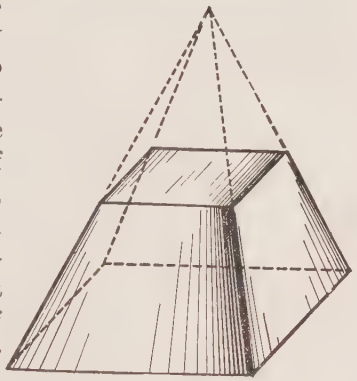


Fig. 139.

Add together the area of the upper base, the lower base, and the square root of the product of the two bases; divide by 3 *and multiply by the altitude of the frustum.*

$$V = \frac{B + b + \sqrt{B \times b}}{3} \times h$$

Example:

A bin is 3 ft. square at the top, 2 ft. square at the bottom, and $2\frac{1}{2}$ ft. deep. What is its volume?

$B = 3^2 = 9$ and $b = 2^2 = 4$

$$V = \frac{B + b + \sqrt{Bb}}{3} \times h = \frac{9 + 4 + \sqrt{9 \times 4}}{3} \times 2.5$$

$$= \frac{9 + 4 + 6}{3} \times 2.5 = 15.8 + \text{cu. ft., } \textit{Answer.}$$

Tapered pins, shafts, etc., are frustums of cones and, therefore, their volumes are calculated as just shown.

93. The Prismoidal Formula.—There are a great many objects encountered in practical work that have two parallel bases and yet they are not prisms or cylinders, neither are they frustums of pyramids or cones. For example, let us take a concrete pier that is 12×30 ft. at the base, 8×24 ft. at the top, and 15 ft. high. This cannot be calculated as a frustum of a pyramid because, while 8 is $\frac{2}{3}$ of 12, 24 is not $\frac{2}{3}$ of 30. Hence, the sides do not all slope toward the same point and this pier is not a frustum of a pyramid. Again, if we take a tub made of half a barrel, we see that it resembles a frustum of a cone, but it "bellies out" in the middle instead of having a straight slant on its side. There is a formula or rule often used for such work that will give very close results for any such object. It can also be used for frustums because it can be reduced to the same form as that given for frustums. The rule reads "*Add together the area of the top base, the lower base, and* 4 *times the sectional area midway between the bases; divide by* 6; *and multiply by the height.*" By adding the two bases and 4 times the midsection, and dividing by 6, we get the section area of a prism that is approximately equivalent in volume, hence the name "prismoidal formula."

Example:

A barrel is 17 in. inside diameter at the heads, 20 in. at the bung, and 26 in. high inside. Calculate its capacity in gallons.

$.7854\times17^2=226.98$ sq. in., area at heads.

$.7854\times20^2=314.16$ sq. in., area at center.

$$V=\frac{226.98+226.98+4\times314.16}{6}\times26$$

$$=\frac{1710.6\times26}{6}=7412.6 \text{ cu. in.}$$

$7412.6\div231=32.1$ gal., *Answer.*

94. The Sphere.—The *Sphere* is the geometrical name for a round or ball-shaped solid. It has been found by measurement, and can be proved by a long process, that the surface of a sphere is just equal to the lateral or curved surface of a cylinder of the same diameter and height, as illustrated in Fig. 140. The circumference of the cylinder is $2\pi r$ and its height is $2r$. Hence, its lateral surface is $2\pi r\times2r=4\pi r^2$. Since $4r^2=d^2$, this can be written $S=\pi d^2$. This is just 4 times the area of a circle of the same diameter as the sphere. Therefore, we can say that the surface of a sphere is just 4 times the area of a circle of the same diameter.

The sphere may be considered as being made up of a great many pyramids (such as shown in Fig. 141) whose bases together form the surface of the sphere and whose apexes meet at the center; since the volume of a pyramid equals $\frac{1}{3}$ the area of base times the height ($=r$ of sphere), and since the sum of the bases

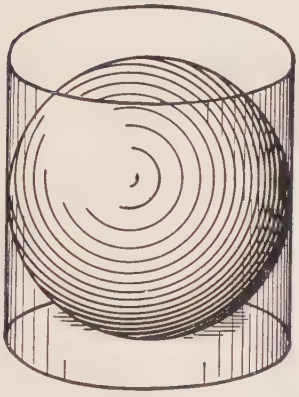


Fig. 140.

Fig. 141.

of all of the pyramids equals the total surface of the sphere ($=4\pi\ r^2$ or $\pi\ d^2$), we have:

$$V=\frac{1}{3}\ (\text{surface})\times(\text{radius}),\ \text{or}$$

$$V=\frac{1}{3}\ (4\pi\ r^2)r=\frac{4}{3}\pi\ r^3,\ \text{or}$$

$$V=\frac{1}{3}\ (\pi\ d^2)\frac{d}{2}=\frac{1}{6}\pi\ d^3.$$

Hence, the volume of a sphere equals $\frac{1}{6}\ \pi$ times the diameter cubed.

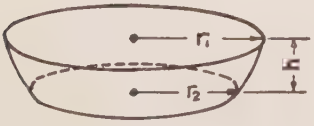


Fig. 142.

Fig. 143.

95. Segment of a Sphere.—A *Segment* of a sphere is a part included between two parallel planes as shown in Fig. 142.

The volume of any body or vessel of this sort is obtained by the formula:

$$V=\frac{h}{2}(\pi\ r_1{}^2+\pi\ r_2{}^2)+\frac{\pi\ h^3}{6}$$

where the letters refer to the dimensions shown in Fig. 142.

A spherical segment of one base is a part of a sphere cut off by a single plane passed through the sphere, as shown in Fig. 143. In this case r_2 of the preceding formula is 0 (zero) and the formula becomes:

$$V = \frac{\pi r_1^2 h}{2} + \frac{\pi h^3}{6}.$$

The area of the curved surface of any segment is equal to $2\pi rh$ where r is the radius of the *sphere* and h is the height of the segment.

Fig. 144.

96. The Ring.—If a ring is formed from stock of any cross-section, the volume of the ring may be obtained by multiplying the area of the cross-section of the stock by the circumference of the circle passing through the center of the cross-section. In a ring made of round stock as shown in Fig. 144, if r represents the radius of the stock and R the radius of the ring, measured at the center of the stock, the volume will be

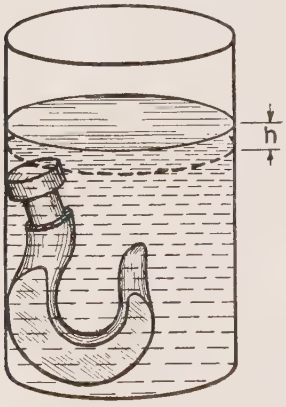


Fig. 145.

$$\pi r^2 \times 2 \pi R \text{ or}$$
$$V = 2 \pi^2 r^2 R$$

The same principle may be applied to any object made by bending a bar into any shape. The volume is the product of the sectional area of the bar times the length of the figure, measured at the center of the stock.

Likewise, the *surface* of any such figure is obtained by multiplying the *circumference* of the stock by the length of the figure measured at the center of the stock. These rules apply to stock of any section, whether round, square, hexagonal, rectangular, or of any other shape.

97. Volumes of Irregular Objects.—When an object is of such an irregular shape that it cannot be divided into regular shapes for calculating its volume, we can weigh it and also weigh a cubic inch or cubic foot of the same material. This will enable us to calculate the volume.

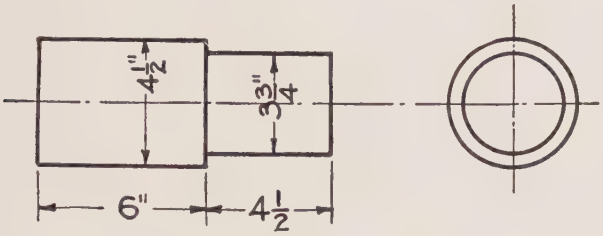


Fig. 146.

Another simple way, if the object will not absorb water, is to sink it in a vessel or tank of water, as shown in Fig. 145. The amount that the water rises indicates the volume of the body. A vessel of uniform section should be used. The area of the surface of the water times the height it rises gives the volume of the object.

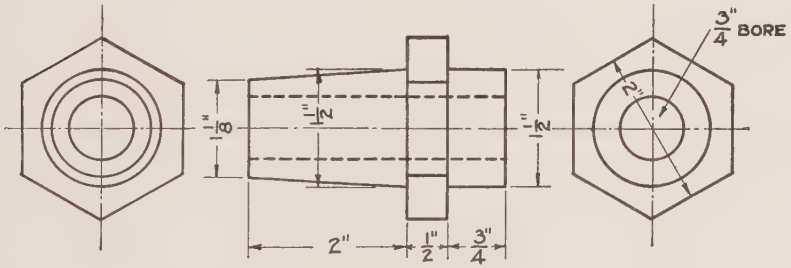


Fig. 147.

For measuring the inside volume of hollow objects, this plan can be reversed, the space being determined from the amount of water required to fill it. This plan is used to measure the clearance space in the cylinders of steam and gas engines.

PROBLEMS

135. Find the weight of the steel crank pin shown in Fig. 146. (Steel = .283 lb. per cubic inch.)

136. Find the weight of the brass push rod guide shown in Fig. 147. (Brass = .301 lb. per cubic inch.)

137. How many ounces of nickel would be used in plating a ball 2 in. in diameter to a depth of $\frac{1}{64}$ in.? (Nickel = 5.14 oz. per cubic inch.)

138. A boiler uses about $\frac{1}{2}$ cu. ft. of water per hour for each horse-power. What size of feed pipe should be used for a 250 H.-P. boiler in order that the water will flow through the pipe at a velocity of 300 ft. per minute?

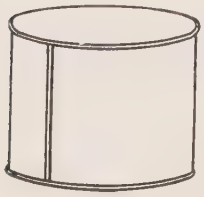


Fig. 148.

139. A gas engine foundation is 6×12 ft. at the bottom, 3×10 ft. at the top, and 5 ft. high. How many cubic feet of concrete are there in it?

140. How many feet of No. 12 steel wire (diam. .109 in.) in a coil weighing 49 lb?

141. Calculate the diameter of a cast iron ball to weigh 16 lb. (Cast iron = .26 lb. per cubic inch.)

142. A cast iron flywheel has an outside diameter of 5 ft. The rim is of rectangular section, 8 in. wide and 3 in. thick. What is the weight of the rim?

143. The vessel in Fig. 145 has an inside diameter of 5 in. When the crane hook is placed in it the water rises $\frac{3}{8}$ in. How many cubic inches of steel are there in the hook?

144. The galvanized iron measure shown in Fig. 148 is 12 in. in diameter and $9\frac{1}{2}$ in. deep and holds approximately $\frac{1}{2}$ bushel. If we allow $1\frac{1}{2}$ in. on the height of the side piece for the roll and lock at the top and bottom, $\frac{3}{4}$ in. on the length of the side piece for the vertical seam, and 1 in. on the diameter of the bottom for the lock, how many square inches of metal would be used in making the measure?

CHAPTER XIV

TRIGONOMETRY—THE TANGENT AND COTANGENT

98. Trigonometry and Its Uses.—Trigonometry is a branch of mathematics that deals with the angles and sides of triangles and their relations to one another. It is a very valuable study for shop men, being especially useful in the study of the milling machine. It is used in the calculations for bevel and spiral gears, in worm and thread milling, in the calculations of the dimensions of screw threads, in taper turning, and in numerous other calculations connected with accurate machine work. It is almost invaluable to the surveyor, the contractor, the draftsman, and, in fact, is used in all sorts of engineering work.

Trigonometry is based on certain "functions" of angles. A *Function* is a quantity that depends on another quantity for its value. A man's pay is a *function* of the time he works and of his rate. Any quantity that depends upon an angle for its value is a function of that angle. If a right triangle is constructed, having a certain angle at one corner, there will be certain definite relations between the sides of this triangle. These ratios are six in number and are called the *trigonometric functions*.

99. The Tangent.—The first function that we will study is called the *tangent*. Let AOB, Fig. 149, be any angle. Remember that the angle is the amount of opening between the lines OA and OB. If from any point P, on one side of the angle, we draw a line PN perpendicular to the other side, we will have a right triangle PNO, with the given angle AOB at one corner of the triangle. Now, no matter where we choose the point P along the line OB, the relation of the side PN to the side ON (that is, the ratio $\frac{PN}{ON}$) will be constant, or unchanged, unless the angle itself is changed.

Take the special case shown in Fig. 150. Here ON is 1 in. and NP is $\frac{3}{4}$ in. Hence, the ratio $\frac{NP}{ON} = \frac{.75}{1.00} = .75$. If at N', 2 in. from O, we erect another perpendicular $N'P'$, it will be

$1\frac{1}{2}$ in. high and the ratio of $\frac{N'P'}{ON'}$ will be $\frac{1.5}{2} = .75$, or the same as before.

In any right triangle, we call the two lines that form the right angle the *sides*, while the line opposite the right angle is called the *hypotenuse*. In speaking of the angle PON in the triangle PON of Fig. 149, the side NP is called the *opposite side*, while the side ON is called the *adjacent side*. The ratio of the opposite side to the adjacent side is called the *Tangent* of the angle. In Fig. 150 the tangent of angle $NOP = \frac{NP}{ON}$; or for short, tan $NOP = \frac{NP}{ON}$. This we found to be .75, hence, for this angle, tan $NOP = \frac{NP}{ON} = .75$. (It should be distinctly understood that the opposite side must be drawn to meet the adjacent side at right angles, or 90°.)

This tangent is a definite quantity for each angle and can only be changed by changing the angle. If we made the angle

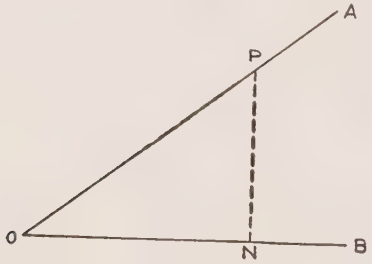


Fig. 149.

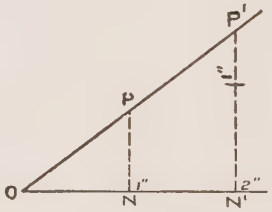


Fig. 150.

greater in either Fig. 149 or 150, PN would be lengthened and the tangent would be increased; if we made the angle less, the tangent would be less. It will be seen, therefore, that if a table of these tangents of angles is available, it offers a simple way of laying out an angle and also of measuring an angle by calculating its tangent and then finding the degrees in a table.

100. To Lay Out An Angle by the Tangent.—It is generally possible to lay out angles more accurately by means of the tangent than to use a protractor.

Suppose that we wish to lay out an angle of 18°. The tan 18° is .3249, as can be seen on page 142 of the tables at the end of this chapter. This means that if a right triangle is drawn

having the ratio of two sides equal to .3249, then one of the angles of the triangle will be 18°. To do this we draw first a base line, which is to be the side of the triangle adjacent to the desired angle.

In Fig. 151 let A be the point where we want the apex of the

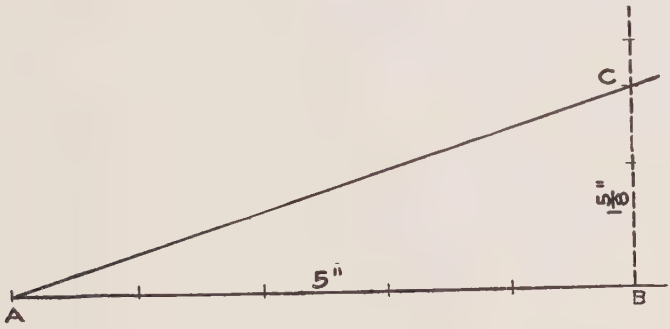


Fig. 151

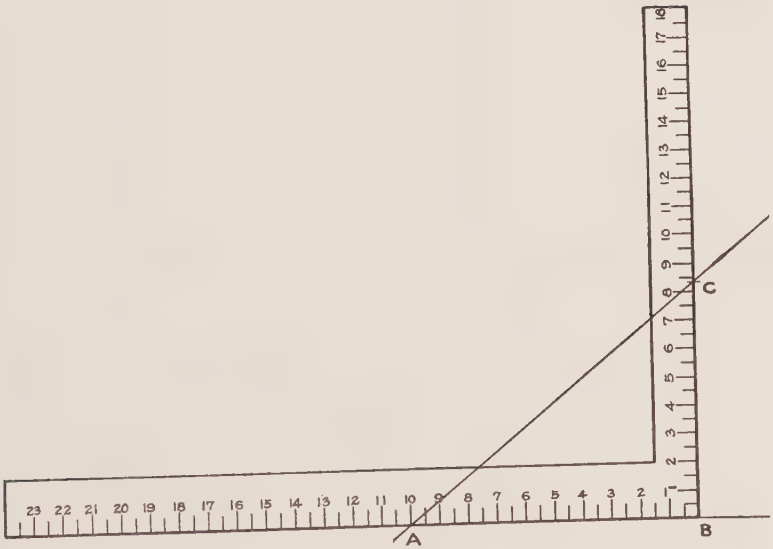


Fig. 152.

angle to lie and let AB be the direction that one side of the angle is to take. We measure off on AB any convenient distance (the longer the distance the more accurate the work will be). Here we have laid off 5 in. on AB. The opposite side of the triangle will lie along a line BC drawn at right angles with AB,

using a square or compasses to get an exact right angle. On BC we must lay off a length such that $\frac{BC}{AB}=.3249$. If AB is made 5 in., then $BC=5\times.3249=1.6245$ in. or practically $1\frac{5}{8}$ in. On BC we mark off $1\frac{5}{8}$ in. from B and through this point we draw a line from A, as shown. Then BAC must be 18° because $\tan BAC=\frac{1.625}{5}=.325$ and .325 is tan 18°.

Angles may in this manner be laid out quite easily by the use of a carpenter's square. A simple plan is to use 10 in. for the length AB and then 10 times the tangent of the angle for the length BC. In Fig. 152 an angle of 40° has been laid out by laying off $AB=10$ in. and marking the point C at 8.39 in. above B (since $\tan 40°=.8391$). A line drawn from C to A will give the desired angle of 40° with AB.

101. To Measure an Angle by its Tangent.—An angle can be measured by its tangent and, from a table of tangents, the number of degrees may be found. Let us measure in this manner the

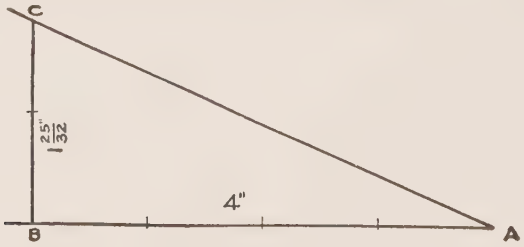


Fig. 153.

angle at A in Fig. 153. On AB we measure off from A some convenient distance, say 4 in. At this distance from A we erect a perpendicular line until it cuts the side AC. The length from B to C of this perpendicular we measure and find to be $1\frac{25}{32}$ or 1.7812+ in.

$$\text{Then, } \tan BAC=\frac{\text{opposite side}}{\text{adjacent side}}=\frac{1.7812+}{4}=.4453$$

From the table on p. 143 it will be seen that .4453 is only one ten-thousandth more than the tangent of 24°. Hence the angle BAC must be practically 24°. The greater the length AB used, the more accurately can the angle be determined.

102. Some Shop Uses of the Tangent.—The tangent is of great value in the laying out or machining of inclined lines or

surfaces. Sometimes the slope of a surface or line is given by line measurements and we need to calculate the slope on degrees; or the slope may be given in degrees and we want to lay it out by line measurements, as was done in Figs. 151 and 152. In such cases a table of tangents is used. Suppose the slanting slot is to be planed in the block shown in Fig. 154. We would probably put the block in a swivel vise on the shaper and swing the vise so that the tool would travel straight through the slot. The vise is graduated in degrees, so we must know the number of degrees in the angle a at which the slot slopes. The slot slopes $5-4=1$ in. in the width of the piece, which is 4 in. Hence, tan $a=1\div 4=.2500$. This is the tangent of a little over 14° (to be exact, 14° $2\frac{1}{4}'$) hence the swivel vise should be turned this number of degrees in order that the shaper may cut the slot at the proper inclination.

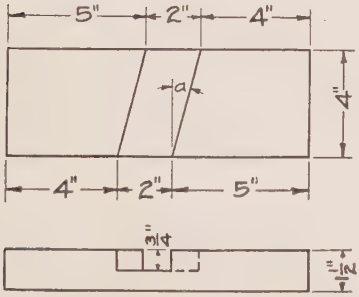


Fig. 154.

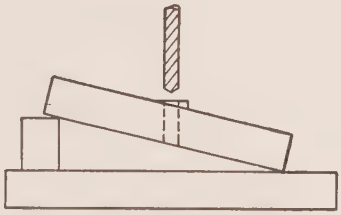


Fig. 155.

Another case is shown in Fig. 155. Here we wish to block up one end of a casting on the drill press table so that the hole will be drilled through at an angle of 12° 40′. The question naturally arises "*how much* shall it be blocked up?" The tangent of 12° 40′ is almost exactly .225. Then, if we put a $2\frac{1}{4}$-in. block under the casting at a horizontal distance of 10 in. from the corner that rests on the table, we will have it blocked up at an angle whose tangent is .225 and this is the desired angle of 12° 40′.

103. Tangents of Some Common Angles.—There are certain angles which occur so frequently in geometric figures that it is well to know how to get their tangents without the tables. Among such angles are 30°, 45°, 60°, and 90°.

The simplest case is the angle of 45°. We know that 45° is

one-half of 90°, and we know that the corner of a square is 90°; so we can work from these facts. We construct the square shown in Fig. 156 and then draw the diagonal OP. NP will be equal to and perpendicular to ON, being sides of the same square.

$$\tan 45° = \frac{NP}{ON} = \frac{\text{one side of square}}{\text{one side of square}} = 1$$

Hence, $\tan 45° = 1$

For angles greater than 45° (and less than 90°) the tangent is greater than 1. For angles between 45° and 0°, the tangent is less than 1.

It is also easy to find the tangent of 30° or of 60°. Let us take the case of a 60° angle. We happen to know that each angle of an equilateral triangle is 60°; we can make use of this fact in finding the tangent.

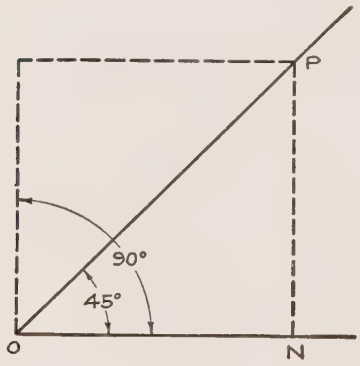


Fig. 156.

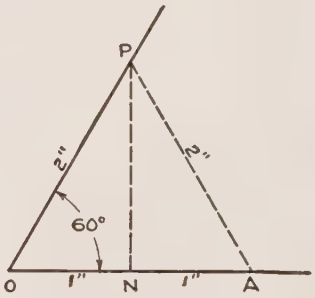


Fig. 157.

For sake of simplicity let us use an equilateral triangle each of whose sides is 2 in. Drawing the triangle as shown in Fig. 157, we know that the angle AOP is 60°. Drop the perpendicular PN upon OA.

$$\tan 60° = \tan AOP = \frac{NP}{ON},$$

but $ON = \frac{1}{2}$ of $OA = 1$,

and $NP = \sqrt{OP^2 - ON^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$

Hence, $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3} = 1.732$.

Using the same figure we can calculate tan 30°. The perpen-

dicular *PN* bisects the angle *OPA*. Hence, the angle *OPN* will be 30°, being half of 60°.

$$\tan 30° = \tan OPN = \frac{ON}{NP} = \frac{1}{\sqrt{3}} = \frac{1}{1.732} = .577$$

The tangent of an angle of 0° is 0. In Fig. 158 suppose the line *OP* to be slowly swung down toward the horizontal line *OA*. Keeping the same adjacent side *ON*, we see that *OP* will cut off shorter and shorter lengths on *NP*, until the opposite side *NP* will become zero when the line *OP* falls exactly on *OA*.

$$\tan 0° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{0}{\text{adjacent side}} = 0.$$

The tangent of 90° is "infinity"; that is, a larger quantity than any quantity we might conceive of. In Fig. 158, suppose we should swing the line *OP* slowly up into a vertical position, meanwhile keeping *NP* just where it is. As *OP* swings up toward the vertical, it would be necessary to extend *NP* farther and farther upward to meet the line *OP* until finally, as *OP* becomes exactly vertical, the two lines would not meet at all. In that case, the length *NP* would be "infinite" and consequently the value of tan 90° is "infinity." The mathematical sign for infinity is ∞. Then, tan 90° is ∞.

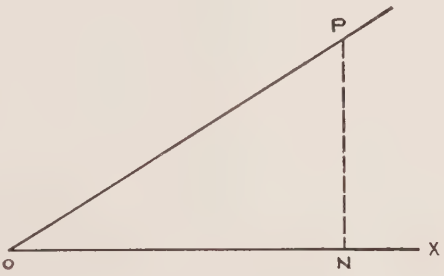


Fig. 158.

Warning.—The student must notice and remember that if we double an angle, the tangent is not double its former value; and if we bisect an angle, we do not cut the tangent in half. There is a definite relation, but it is complicated and it is better to use the table in all cases. Notice that tan 90° = , ∞ but tan 45° = 1, also that tan 60° = 1.732+, but tan 30° = .577+. These cases show that there is apparently no simple relation between the tangent of an angle and the tangent of half the angle.

104. The Cotangent.—If we take any angle such as *POX* of Fig. 158 and drop the perpendicular *PN*, we have the right angle *NOP*. Now, it is well known that the sum of three angles of any triangle = 180°. Hence, the sum of the three angles in the triangle *NOP* will be 180°; but the angle *ONP* is a right angle or

90° by construction. Then the sum of the other two angles must equal the remaining 90°.

$$\begin{aligned} \text{Angle } NOP + \text{angle } OPN + \text{angle } ONP &= 180° \\ NOP + \quad OPN + \quad 90° &= 180° \\ NOP + \quad OPN = 180° - 90° &= 90° \end{aligned}$$

From these equations it will be seen that the sum of the two acute angles of any right triangle equals 90°; in other words, the acute angles of any right angle triangle are *complements* of each other.

(For definition of "acute angle" and "complement" or "complementary angle," see Chapter X.)

Keeping in mind that angles *NOP* and *OPN* are complementary, or that either may be considered as the complement of the other, let us see the result:

We have dealt with the ratio, $\frac{\text{opposite}}{\text{adjacent}}$; now consider the reverse ratio, $\frac{\text{adjacent}}{\text{opposite}}$, or $\frac{ON}{NP}$.

This ratio is called the *Cotangent* of *NOP*, or

$$\cot NOP = \frac{ON}{NP} = \frac{\text{adjacent}}{\text{opposite}}$$

The word cotangent is an abbreviation or shortening of the words: "complementary tangent" or "tangent of the complementary angle." This comes from the fact that:

$\cot NOP = \frac{\text{adjacent}}{\text{opposite}} = \frac{ON}{NP}$, and that $\frac{ON}{NP}$ is also the tan *OPN*, the complementary angle of *NOP*. The reason for having and using the cotangent is that it is usually considered easier to multiply than to divide; so, in a problem where we would have to divide by the tangent, we can, instead, use the cotangent and multiply.

Notice that:

$$\cot 60° = \frac{1}{\tan 60°} = \tan 30° = .5774$$

$$\cot 30° = \frac{1}{\tan 30°} = \tan 60° = 1.7321$$

$$\cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1.$$

The tangent of any angle is the cotangent of its complement, and the cotangent of any angle is the tangent of its complement.

To illustrate:
tan 40° = .8391 = cot 50° (40° and 50° are complementary)
tan 50° = 1.1918 = cot 40°
tan 25° = .4663 = cot 65° (25° and 65° are complementary)
tan 65° = 2.1445 = cot 25°

105. Use of the Tables.—Now turn to the table headed "Natural Functions of Angles," which will be found on page 139.

In the first column, headed °, meaning degrees, will be found angles from 0° to 45°; in the second column, headed ′, meaning minutes, are given the numbers of minutes between the degrees given in the first column; in the fifth column, headed tangent, will be found tangents of the angles from 0° to 45°.

In the last column will be found angles from 90° to 45°, each angle in the last column being the complement of the angle in the first column, and *vice versa*. *With the angles found in the last column, use the words at the bottom of the page.*

Example:
Find tan 70°.
Find 70° in the last column; look at the *bottom* of page for column labeled *tangent;* in this column opposite 70° is 2.7475.
Then, tan 70° = 2.7475.
You will notice also that this tan 70° is the same as cot 20°, as indicated at the top and left.

The cotangents are determined in the same way, the values being in the sixth column for angles up to 45°, the degrees and minutes being at the left side of the pages. For angles greater than 45° use the degrees and minutes at the right side of the pages and use the column labels at the bottoms of the pages.

Examples:

cot 17° = 3.2709	cot 60° = .5774
cot 31° 40′ = 1.6212	cot 73° 30′ = .2962

The other columns are explained in the chapters following.

106. Bevel Gears.—When two shafts meet at 90° and are connected by bevel gears, the teeth must slope toward the intersection of the center lines of the shafts. The angle *PON* in Fig. 159 is called the "pitch angle" of the gear *A*. Likewise, the angle *POM* is the pitch angle of gear *B*. Suppose the gears *A* and *B* are 30 and 20 in. in diameter, respectively. Then *PN* will be 15 in. and *PM* or *NO*, 10 in.

$$\tan PON = \frac{PN}{ON} = \frac{15}{10} = 1.5$$

Hence, PON is between 56° 10′ and 56° 20′ or, to be more exact, is 56° 18′. The gear A must therefore be cut so that its pitch angle is 56° 18′. Likewise, the pitch angle of B is found to be 33° 42′, since $\tan POM = \frac{10}{15} = .6667$, from which $POM = 33°\ 42'$. POM could also be found from its being the complement of PON.

$$POM = 90° - PON = 90° - 56°\ 18' = 33°\ 42'$$

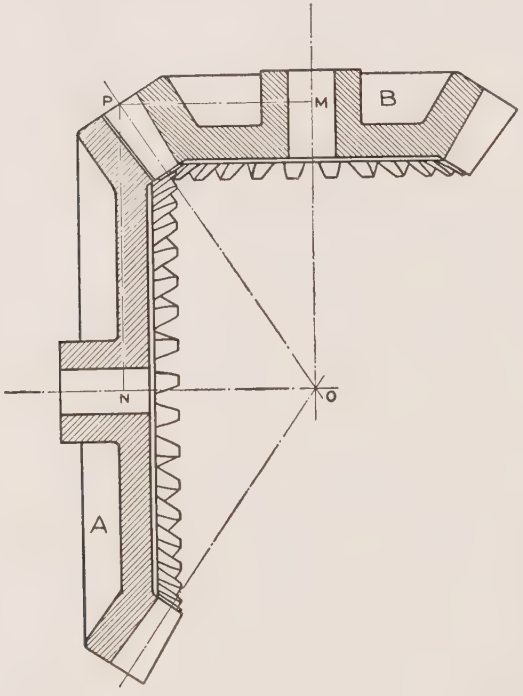


Fig. 159

The teeth on A would have a pitch angle of 56° 18′ with the center line of the gear, while the teeth on B would have a pitch angle of 33° 42′ with the center line of that gear.

PROBLEMS

145. Find from the table the following tangents: tan 20°; tan 70°; tan 25°; tan 22½°; tan 67° 20′.

146. Find from the table the following: cot 5°; cot 70°; cot 14° 30′; cot 67° 30′; cot 34° 40′.

147. Copy the angle of Fig. 160 by the method explained in Chapter X. Then from some point on OB erect a line perpendicular to OB until

it cuts OA. Scale the necessary lines and calculate tan AOB. Then look in the table and find the number of degrees in angle AOB.

148. A hill road rises $1\frac{1}{2}$ ft. vertically for each 12 ft. measured horizontally. Find the angle of the road with the horizontal.

149. Fig. 161 shows a regular octagon being milled out of a solid round bar. The "short diameter" of the octagon is to be $1\frac{1}{2}$ in., as shown. Calcu-

Fig. 160.

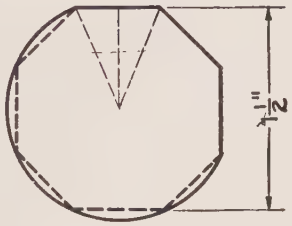


Fig. 161.

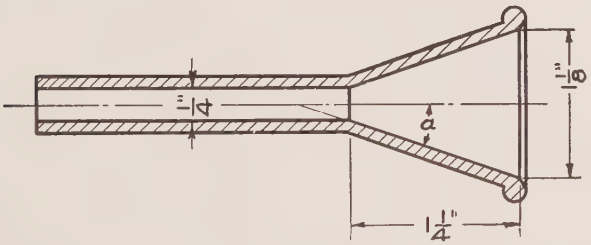


Fig. 162.

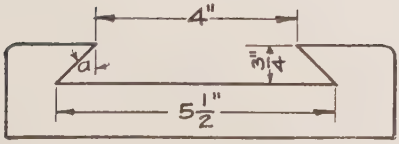


Fig. 163.

late the width of one face of the octagon. The dotted triangles will suggest the method of calculation.

150. Calculate the angle a at which a tool must be fed to bore out the mouthpiece or funnel shown in Fig. 162.

151. Construct an angle of 14° 30′ by means of its tangent.

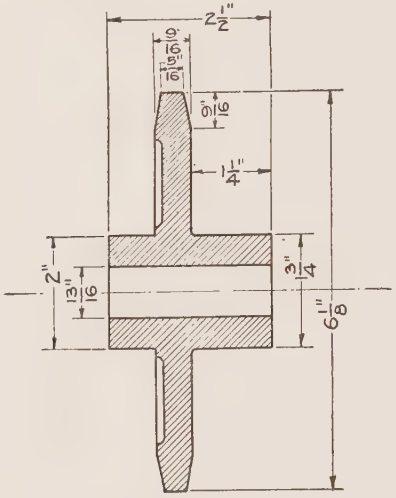


Fig. 164.

152. The dove-tail slot shown in Fig. 163 is 4 in. wide at the top, $5\frac{1}{2}$ in. wide at the bottom, and $\frac{3}{4}$ in. deep. Calculate the angle at which the sides of the slot slope from the vertical.

153. Fig. 164 shows a section of a sprocket wheel. From the given dimensions calculate the angle at which the sides of the teeth are bevelled off.

154. A bevel gear 8 in. in diameter is measured and found to have a pitch angle of 58°. What must be the diameter and pitch angle of the gear to mesh with it, if the gears are to connect two shafts at an angle of 90°?

NATURAL FUNCTIONS OF ANGLES

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
0	0	.0000	Infinite	.0000	Infinite	1.0000	1.0000	0	90
	10	.0029	343.7752	.0029	343.7737	1.0000	1.0000	50	
	20	.0058	171.8883	.0058	171.8854	1.0000	1.0000	40	
	30	.0087	114.5930	.0087	114.5887	1.0000	1.0000	30	
	40	.0116	85.9456	.0116	85.9398	1.0001	.9999	20	
	50	.0145	68.7574	.0145	68.7501	1.0001	.9999	10	
1	0	.0175	57.2987	.0175	57.2900	1.0002	.9998	0	89
	10	.0204	49.1141	.0204	49.1039	1.0002	.9998	50	
	20	.0233	42.9757	.0233	42.9641	1.0003	.9997	40	
	30	.0262	38.2016	.0262	38.1885	1.0003	.9997	30	
	40	.0291	34.3823	.0291	34.3678	1.0004	.9996	20	
	50	.0320	31.2576	.0320	31.2416	1.0005	.9995	10	
2	0	.0349	28.6537	.0349	28.6363	1.0006	.9994	0	88
	10	.0378	26.4505	.0378	26.4316	1.0007	.9993	50	
	20	.0407	24.5621	.0407	24.5418	1.0008	.9992	40	
	30	.0436	22.9256	.0437	22.9038	1.0009	.9990	30	
	40	.0465	21.4937	.0466	21.4704	1.0011	.9989	20	
	50	.0494	20.2303	.0495	20.2056	1.0012	.9988	10	
3	0	.0523	19.1073	.0524	19.0811	1.0014	.9986	0	87
	10	.0552	18.1026	.0553	18.0750	1.0015	.9985	50	
	20	.0581	17.1984	.0582	17.1693	1.0017	.9983	40	
	30	.0610	16.3804	.0612	16.3499	1.0019	.9981	30	
	40	.0640	15.6368	.0641	15.6048	1.0021	.9974	20	
	50	.0669	14.9579	.0670	14.9244	1.0022	.9978	10	
4	0	.0698	14.3356	.0699	14.3007	1.0024	.9976	0	86
	10	.0727	13.7631	.0729	13.7267	1.0027	.9974	50	
	20	.0756	13.2347	.0758	13.1969	1.0029	.9971	40	
	30	.0785	12.7455	.0787	12.7062	1.0031	.9969	30	
	40	.0814	12.2913	.0816	12.2505	1.0033	.9967	20	
	50	.0843	11.8684	.0846	11.8262	1.0036	.9964	10	
5	0	.0872	11.4737	.0875	11.4301	1.0038	.9962	0	85
	10	.0901	11.1045	.0904	11.0594	1.0041	.9959	50	
	20	.0930	10.7585	.0934	10.7119	1.0044	.9957	40	
	30	.0958	10.4334	.0963	10.3854	1.0046	.9954	30	
	40	.0987	10.1275	.0992	10.0780	1.0049	.9951	20	
	50	.1016	9.8391	.1022	9.7882	1.0052	.9948	10	84
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 84° 10′ to 90° read from bottom of table upward.

NATURAL FUNCTIONS OF ANGLES

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
6	0	.1045	9.5668	.1051	9.5144	1.0055	.9945	0	84
	10	.1074	9.3092	.1080	9.2553	1.0058	.9942	50	
	20	.1103	9.0652	.1110	9.0098	1.0061	.9939	40	
	30	.1132	8.8337	.1139	8.7769	1.0065	.9936	30	
	40	.1161	8.6138	.1169	8.5555	1.0068	.9932	20	
	50	.1190	8.4046	.1198	8.3450	1.0072	.9929	10	
7	0	.1219	8.2055	.1228	8.1443	1.0075	.9925	0	83
	10	.1248	8.0156	.1257	7.9530	1.0079	.9922	50	
	20	.1276	7.8344	.1287	7.7704	1.0083	.9918	40	
	30	.1305	7.6613	.1317	7.5958	1.0086	.9914	30	
	40	.1334	7.4957	.1346	7.4287	1.0090	.9911	20	
	50	.1363	7.3372	.1376	7.2687	1.0094	.9907	10	
8	0	.1392	7.1853	.1405	7.1154	1.0098	.9903	0	82
	10	.1421	7.0396	.1435	6.9682	1.0102	.9899	50	
	20	.1449	6.8998	.1465	6.8270	1.0107	.9894	40	
	30	.1478	6.6755	.1495	6.6912	1.0111	.9890	30	
	40	.1507	6.6363	.1524	6.5606	1.0116	.9886	20	
	50	.1536	6.5121	.1554	6.4348	1.0120	.9881	10	
9	0	.1564	6.3924	.1584	6.3138	1.0125	.9877	0	81
	10	.1593	6.2772	.1614	6.1970	1.0129	.9872	50	
	20	.1622	6.1661	.1644	6.0844	1.0134	.9868	40	
	30	.1650	6.0589	.1673	5.9758	1.0139	.9863	30	
	40	.1679	5.9554	.1703	5.8708	1.0144	.9858	20	
	50	.1708	5.8554	.1733	5.7694	1.0149	.9853	10	
10	0	.1736	5.7588	.1763	5.6713	1.0154	.9848	0	80
	10	.1765	5.6653	.1793	5.5764	1.0160	.9843	50	
	20	.1794	5.5749	.1823	5.4845	1.0165	.9838	40	
	30	.1822	5.4874	.1853	5.3955	1.0170	.9833	30	
	40	.1851	5.4026	.1884	5.3093	1.0176	.9827	20	
	50	.1880	5.3205	.1914	5.2257	1.0182	.9822	10	
11	0	.1908	5.2408	.1944	5.1446	1.0187	.9816	0	79
	10	.1937	5.1636	.1974	5.0658	1.0193	.9811	50	
	20	.1965	5.0886	.2004	4.9894	1.0199	.9805	40	
	30	.1994	5.0158	.2035	4.9152	1.0205	.9799	30	
	40	.2022	4.9452	.2065	4.8430	1.0211	.9793	20	
	50	.2051	4.8765	.2095	4.7729	1.0217	.9787	10	78
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 78° 10′ to 84° read from bottom of table upward.

NATURAL FUNCTIONS OF ANGLES.—*Continued*

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
12	0	.2079	4.8097	.2126	4.7046	1.0223	.9781	0	78
	10	.2108	4.7448	.2156	4.6382	1.0230	.9775	50	
	20	.2136	4.6817	.2186	4.5736	1.0236	.9769	40	
	30	.2164	4.6202	.2217	4.5107	1.0243	.9763	30	
	40	.2193	4.5604	.2247	4.4494	1.0249	.9757	20	
	50	.2221	4.5022	.2278	4.3897	1.0256	.9750	10	
13	0	.2250	4.4454	.2309	4.3315	1.0263	.9744	0	77
	10	.2278	4.3901	.2339	4.2747	1.0270	.9737	50	
	20	.2306	4.3362	.2370	4.2193	1.0277	.9730	40	
	30	.2334	4.2837	.2401	4.1653	1.0284	.9724	30	
	40	.2363	4.2324	.2432	4.1126	1.0291	.9717	20	
	50	.2391	4.1824	.2462	4.0611	1.0299	.9710	10	
14	0	.2419	4.1336	.2493	4.0108	1.0306	.9703	0	76
	10	.2447	4.0859	.2524	3.9617	1.0314	.9696	50	
	20	.2476	4.0394	.2555	3.9136	1.0321	.9689	40	
	30	.2504	3.9939	.2586	3.8667	1.0329	.9681	30	
	40	.2532	3.9495	.2617	3.8208	1.0336	.9674	20	
	50	.2560	3.9061	.2648	3.7760	1.0345	.9667	10	
15	0	.2588	3.8637	.2679	3.7321	1.0353	.9659	0	75
	10	.2616	3.8222	.2711	3.6891	1.0361	.9652	50	
	20	.2644	3.7817	.2742	3.6470	1.0369	.9644	40	
	30	.2672	3.7420	.2773	3.6059	1.0377	.9636	30	
	40	.2700	3.7032	.2805	3.5656	1.0386	.9628	20	
	50	.2728	3.6652	.2836	3.5261	1.0394	.9621	10	
16	0	.2756	3.6280	.2867	3.4874	1.0403	.9613	0	74
	10	.2784	3.5915	.2899	3.4495	1.0412	.9605	50	
	20	.2812	3.5559	.2931	3.4124	1.0421	.9596	40	
	30	.2840	3.5209	.2962	3.3759	1.0430	.9588	30	
	40	.2868	3.4867	.2994	3.3402	1.0439	.9580	20	
	50	.2896	3.4532	.3026	3.3052	1.0448	.9572	10	
17	0	.2924	3.4203	.3057	3.2709	1.0457	.9563	0	73
	10	.2952	3.3881	.3089	3.2371	1.0466	.9555	50	
	20	.2979	3.3565	.3121	3.2041	1.0476	.9546	40	
	30	.3007	3.3255	.3153	3.1716	1.0485	.9537	30	
	40	.3035	3.2951	.3185	3.1397	1.0495	.9528	20	
	50	.3062	3.2653	.3217	3.1084	1.0505	.9520	10	72
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 72° 10′ to 78° read from bottom upward.

NATURAL FUNCTIONS OF ANGLES.—*Continued*

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
18	0	.3090	3.2361	.3249	3.0777	1.0515	.9511	0	72
	10	.3118	3.2074	.3281	3.0475	1.0525	.9502	50	
	20	.3145	3.1792	.3314	3.0178	1.0535	.9492	40	
	30	.3173	3.1515	.3346	2.9887	1.0545	.9483	30	
	40	.3201	3.1244	.3378	2.9600	1.0555	.9474	20	
	50	.3228	3.0977	.3411	2.9319	1.0566	.9465	10	
19	0	.3256	3.0716	.3443	2.9042	1.0576	.9455	0	71
	10	.3283	3.0458	.3476	2.8770	1.0587	.9446	50	
	20	.3311	3.0206	.3508	2.8502	1.0598	.9436	40	
	30	.3338	2.9957	.3541	2.8239	1.0609	.9426	30	
	40	.3365	2.9713	.3574	2.7980	1.0620	.9417	20	
	50	.3393	2.9474	.3607	2.7725	1.0631	.9407	10	
20	0	.3420	2.9238	.3640	2.7475	1.0642	.9397	0	70
	10	.3448	2.9006	.3673	2.7228	1.0653	.9387	50	
	20	.3475	2.8779	.3706	2.6985	1.0665	.9377	40	
	30	.3502	2.8555	.3739	2.6746	1.0676	.9367	30	
	40	.3529	2.8334	.3772	2.6511	1.0688	.9357	20	
	50	.3557	2.8117	.3805	2.6279	1.0700	.9346	10	
21	0	.3584	2.7904	.3839	2.6051	1.0712	.9336	0	69
	10	.3611	2.7695	.3872	2.5826	1.0724	.9325	50	
	20	.3638	2.7488	.3906	2.5605	1.0736	.9315	40	
	30	.3665	2.7285	.3939	2.5386	1.0748	.9304	30	
	40	.3692	2.7085	.3973	2.5172	1.0760	.9293	20	
	50	.3719	2.6888	.4006	2.4960	1.0773	.9283	10	
22	0	.3746	2.6695	.4040	2.4751	1.0785	.9272	0	68
	10	.3773	2.6504	.4074	2.4545	1.0798	.9261	50	
	20	.3800	2.6316	.4108	2.4342	1.0811	.9250	40	
	30	.3827	2.6131	.4142	2.4142	1.0824	.9239	30	
	40	.3854	2.5949	.4176	2.3945	1.0837	.9228	20	
	50	.3881	2.5770	.4210	2.3750	1.0850	.9216	10	
23	0	.3907	2.5593	.4245	2.3559	1.0864	.9205	0	67
	10	.3934	2.5419	.4279	2.3369	1.0877	.9194	50	
	20	.3961	2.5247	.4314	2.3183	1.0891	.9182	40	
	30	.3987	2.5078	.4348	2.2998	1.0904	.9171	30	
	40	.4014	2.4912	.4383	2.2817	1.0918	.9159	20	
	50	.4041	2.4748	.4417	2.2637	1.0932	.9147	10	66
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 66° 10′ to 72° read from bottom of table upward.

NATURAL FUNCTIONS OF ANGLES.—*Continued*

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
24	0	.4067	2.4586	.4452	2.2460	1.0946	.9135	0	66
	10	.4094	2.4426	.4487	2.2286	1.0961	.9124	50	
	20	.4120	2.4269	.4522	2.2113	1.0975	.9112	40	
	30	.4147	2.4114	.4557	2.1943	1.0990	.9100	30	
	40	.4173	2.3961	.4592	2.1775	1.1004	.9088	20	
	50	.4200	2.3811	.4628	2.1609	1.1019	.9075	10	
25	0	.4226	2.3662	.4663	2.1445	1.1034	.9063	0	65
	10	.4253	2.3515	.4699	2.1283	1.1049	.9051	50	
	20	.4279	2.3371	.4734	2.1123	1.1064	.9038	40	
	30	.4305	2.3228	.4770	2.0965	1.1079	.9026	30	
	40	.4331	2.3088	.4806	2.0809	1.1095	.9013	20	
	50	.4358	2.2949	.4841	2.0655	1.1110	.9001	10	
26	0	.4388	2.2812	.4877	2.0503	1.1126	.8988	0	64
	10	.4410	2.2677	.4913	2.0353	1.1142	.8975	50	
	20	.4436	2.2543	.4950	2.0204	1.1158	.8962	40	
	30	.4462	2.2412	.4986	2.0057	1.1174	.8949	30	
	40	.4488	2.2282	.5022	1.9912	1.1190	.8936	20	
	50	.4514	2.2153	.5059	1.9769	1.1207	.8923	10	
27	0	.4540	2.2027	.5095	1.9626	1.1223	.8910	0	63
	10	.4566	2.1902	.5132	1.9486	1.1240	.8897	50	
	20	.4592	2.1779	.5169	1.9347	1.1257	.8884	40	
	30	.4617	2.1657	.5206	1.9210	1.1274	.8870	30	
	40	.4643	2.1537	.5243	1.9074	1.1291	.8857	20	
	50	.4669	2.1418	.5280	1.8940	1.1308	.8843	10	
28	0	.4695	2.1301	.5317	1.8807	1.1326	.8829	0	62
	10	.4720	2.1185	.5355	1.8676	1.1343	.8816	50	
	20	.4746	2.1070	.5392	1.8546	1.1361	.8802	40	
	30	.4772	2.0957	.5430	1.8417	1.1379	.8788	30	
	40	.4797	2.0846	.5467	1.8291	1.1397	.8774	20	
	50	.4823	2.0736	.5505	1.8165	1.1415	.8760	10	
29	0	.4848	2.0627	.5543	1.8040	1.1434	.8746	0	61
	10	.4874	2.0519	.5581	1.7917	1.1452	.8732	50	
	20	.4899	2.0413	.5619	1.7796	1.1471	.8718	40	
	30	.4924	2.0308	.5658	1.7675	1.1490	.8704	30	
	40	.4950	2.0204	.5696	1.7556	1.1509	.8689	20	
	50	.4975	2.0101	.5735	1.7437	1.1528	.8675	10	60
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 60° 10′ to 66° read from bottom of table upward.

NATURAL FUNCTIONS OF ANGLES.—*Continued*

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
30	0	.5000	2.0000	.5774	1.7321	1.1547	.8660	0	60
	10	.5025	1.9900	.5812	1.7205	1.1567	.8646	50	
	20	.5050	1.9801	.5851	1.7090	1.1586	.8631	40	
	30	.5075	1.9703	.5890	1.6977	1.1606	.8616	30	
	40	.5100	1.9606	.5930	1.6864	1.1626	.8601	20	
	50	.5125	1.9511	.5969	1.6753	1.1646	.8587	10	
31	0	.5150	1.9416	.6009	1.6643	1.1666	.8572	0	59
	10	.5175	1.9323	.6048	1.6534	1.1687	.8557	50	
	20	.5200	1.9230	.6088	1.6426	1.1708	.8542	40	
	30	.5225	1.9139	.6128	1.6319	1.1728	.8526	30	
	40	.5250	1.9048	.6168	1.6212	1.1749	.8511	20	
	50	.5275	1.8959	.6208	1.6107	1.1770	.8496	10	
32	0	.5299	1.8871	.6249	1.6003	1.1792	.8480	0	58
	10	.5324	1.8783	.6289	1.5900	1.1813	.8465	50	
	20	.5348	1.8697	.6330	1.5798	1.1835	.8450	40	
	30	.5373	1.8612	.6371	1.5697	1.1857	.8434	30	
	40	.5398	1.8527	.6412	1.5597	1.1879	.8418	20	
	50	.5422	1.8443	.6453	1.5497	1.1901	.8403	10	
33	0	.5446	1.8361	.6494	1.5399	1.1924	.8387	0	57
	10	.5471	1.8279	.6535	1.5301	1.1946	.8371	50	
	20	.5495	1.8198	.6577	1.5204	1.1969	.8355	40	
	30	.5519	1.8118	.6619	1.5108	1.1992	.8339	30	
	40	.5544	1.8039	.6661	1.5013	1.2015	.8323	20	
	50	.5568	1.7960	.6703	1.4919	1.2039	.8307	10	
34	0	.5592	1.7883	.6745	1.4826	1.2062	.8290	0	56
	10	.5616	1.6806	.6787	1.4733	1.2086	.8274	50	
	20	.5640	1.7730	.6830	1.4641	1.2110	.8258	40	
	30	.5664	1.7655	.6873	1.4550	1.2134	.8241	30	
	40	.5688	1.7581	.6916	1.4460	1.2158	.8225	20	
	50	.5712	1.7507	.6959	1.4370	1.2183	.8208	10	
35	0	.5736	1.7434	.7002	1.4281	1.2208	.8192	0	55
	10	.5760	1.7362	.7046	1.4193	1.2233	.8175	50	
	20	.5783	1.7291	.7089	1.4106	1.2258	.8158	40	
	30	.5807	1.7221	.7133	1.4019	1.2283	.8141	30	
	40	.5831	1.7151	.7177	1.3934	1.2309	.8124	20	
	50	.5854	1.7081	.7221	1.3848	1.2335	.8107	10	54
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 54° 10′ to 60° read from bottom of table upward.

NATURAL FUNCTIONS OF ANGLES.—*Continued*

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
36	0	.5878	1.7013	.7265	1.3764	1.2361	.8090	0	54
	10	.5901	1.6945	.7310	1.3680	1.2387	.8073	50	
	20	.5925	1.6878	.7355	1.3597	1.2413	.8056	40	
	30	.5948	1.6812	.7400	1.3514	1.2440	.8039	30	
	40	.5972	1.6746	.7445	1.3432	1.2467	.8021	20	
	50	.5995	1.6681	.7490	1.3351	1.2494	.8004	10	
37	0	.6018	1.6616	.7536	1.3270	1.2521	.7986	0	53
	10	.6041	1.6553	.7581	1.3190	1.2549	.7969	50	
	20	.6065	1.6489	.7627	1.3111	1.2577	.7951	40	
	30	.6088	1.6427	.7676	1.3032	1.2605	.7934	30	
	40	.6111	1.6365	.7720	1.2954	1.2633	.7916	20	
	50	.6134	1.6303	.7766	1.2876	1.2662	.7898	10	
38	0	.6157	1.6243	.7813	1.2799	1.2690	.7880	0	52
	10	.6180	1.6183	.7860	1.2723	1.2719	.7862	50	
	20	.6202	1.6123	.7907	1.2647	1.2748	.7844	40	
	30	.6225	1.6064	.7954	1.2572	1.2778	.7826	30	
	40	.6248	1.6005	.8002	1.2497	1.2808	.7808	20	
	50	.6271	1.5948	.8041	1.2423	1.2837	.7790	10	
39	0	.6293	1.5890	.8098	1.2349	1.2868	.7771	0	51
	10	.6316	1.5833	.8146	1.2276	1.2898	.7753	50	
	20	.6338	1.5777	.8195	1.2203	1.2929	.7735	40	
	30	.6361	1.5721	.8243	1.2131	1.2960	.7716	30	
	40	.6383	1.5666	.8292	1.2059	1.2991	.7698	20	
	50	.6406	1.5611	.8342	1.1988	1.3022	.7679	10	
40	0	.6428	1.5557	.8391	1.1918	1.3054	.7660	0	50
	10	.6450	1.5504	.8441	1.1847	1.3086	.7642	50	
	20	.6472	1.5450	.8491	1.1778	1.3118	.7623	40	
	30	.6494	1.5398	.8541	1.1708	1.3151	.7604	30	
	40	.6517	1.5345	.8591	1.1640	1.3184	.7585	20	
	50	.6539	1.5294	.8642	1.1571	1.3217	.7566	10	
41	0	.6561	1.5243	.8693	1.1504	1.3250	.7547	0	49
	10	.6583	1.5192	.8744	1.1436	1.3284	.7528	50	
	20	.6604	1.5141	.8796	1.1369	1.3318	.7509	40	
	30	.6626	1.5092	.8847	1.1303	1.3352	.7490	30	
	40	.6648	1.5042	.8899	1.1237	1.3386	.7470	20	
	50	.6670	1.4993	.8952	1.1171	1.3421	.7451	10	48
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 48° 10′ to 54° read from bottom of table upward.

NATURAL FUNCTIONS OF ANGLES.—*Continued*

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
42	0	.6691	1.4945	.9004	1.1106	1.3456	.7431	0	48
	10	.6713	1.4897	.9057	1.1041	1.3492	.7412	50	
	20	.6734	1.4849	.9110	1.0977	1.3527	.7392	40	
	30	.6756	1.4802	.9163	1.0913	1.3563	.7373	30	
	40	.6777	1.4755	.9217	1.0850	1.3600	.7353	20	
	50	.6799	1.4709	.9271	1.0786	1.3636	.7333	10	
43	0	.6820	1.4663	.9325	1.0724	1.3673	.7314	0	47
	10	.6841	1.4617	.9380	1.0661	1.3711	.7294	50	
	20	.6862	1.4572	.9435	1.0599	1.3748	.7274	40	
	30	.6884	1.4527	.9490	1.0538	1.3786	.7254	30	
	40	.6905	1.4483	.9545	1.0477	1.3824	.7234	20	
	50	.6926	1.4439	.9501	1.0416	1.3863	.7214	10	
44	0	.6947	1.4396	.9657	1.0355	1.3902	.7193	0	46
	10	.6967	1.4352	.9713	1.0295	1.3941	.7173	50	
	20	.6988	1.4310	.9770	1.0235	1.3980	.7153	40	
	30	.7009	1.4267	.9827	1.0176	1.4020	.7133	30	
	40	.7030	1.4225	.9884	1.0117	1.4061	.7112	20	
	50	.7050	1.4183	.9942	1.0058	1.4101	.7092	10	
45	0	.7071	1.4142	1.0000	1.0000	1.4142	.7071	0	45
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 45° to 48° read from bottom of table upward.

CHAPTER XV

SOME USES OF THE TANGENT AND COTANGENT

107. Calculating Heights and Distances.—The height of a chimney, the width of a river, or any other distance which cannot be conveniently measured, can be readily determined by the application of a little trigonometry.

Let us imagine that we want to measure the distance from the point B to the point C across the river shown in Fig 165. On

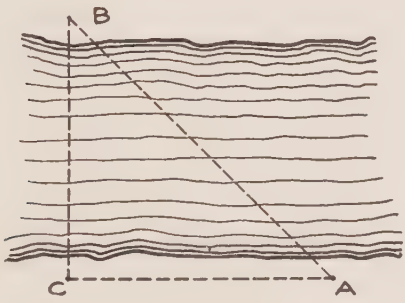


Fig. 165.

one bank of the stream we can measure off any convenient distance, AC, in a direction at right angles to the line BC. BC and AC will be two sides of a right triangle of which the hypotenuse is the line AB. If we measure the angle at A (angle BAC), we can calculate the distance BC, because the ratio of BC to AC is the tangent of the angle BAC.

$$\frac{BC}{AC} = \tan BAC$$

$$\text{or } BC = AC \times \tan BAC$$

The best and most accurate way to measure the angle at A is to use a surveyor's transit. This consists of a tripod surmounted by a sort of telescope which can be swung into any position. The transit can be set up at A and sighted on C. The angle through which it must then be swung to sight it on B can be observed on

the instrument. This will be the desired angle BAC. The transit can also be used at C to get an exact right angle when laying out the line CA.

Example:

A bridge is to be constructed with the end piers at the points C and B, Fig. 165. A distance CA of 600 ft. is laid off from C at right angles to BC. The transit is then set up at A and the angle BAC found to be 37° 20′. What will be the length of the bridge from C to B?

$$BC = AC \tan 37° 20'$$
$$\tan 37° 20' = .7627 \text{ and } AC = 600$$
$$BC = 600 \times .7627 = 457.62 \text{ ft., } \textit{Answer.}$$

The same process can be used for measuring the heights of buildings, chimneys, etc., by laying off some certain distance AC, Fig. 166, on the horizontal and then with the transit set up at the point A, determining the "angle of elevation," as it

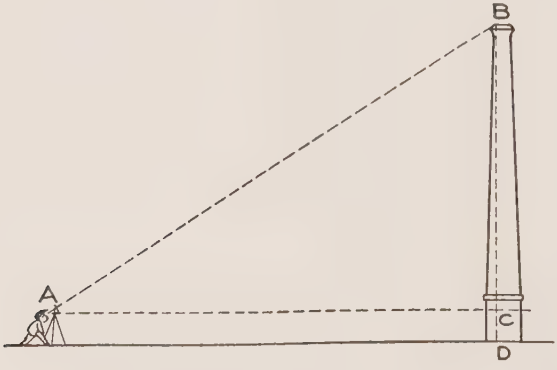


Fig. 166.

is called, of the chimney. Then $BC = AC \tan BAC$. After calculating BC we must add the height CD, which will be the same as the height of the transit if it is set up on a level with the base of the chimney.

Example:

At a distance of 200 ft., the angle of elevation of the chimney in Fig. 166 is 20°, and the transit is 5 ft. above the level of the base of the chimney. What is the total height of the chimney?

$$BC = AC \tan 20°$$
$$\tan 20° = .3640 \text{ and } AC = 200 \text{ ft.}$$
$$BC = 200 \times .364 = 72.8 \text{ ft.}$$
$$BD = BC + CD = 72.8 + 5 = 77.8 \text{ ft., } \textit{Answer.}$$

A simple method that will give approximate results without the use of a transit is to use a draftsman's triangle as shown in Fig. 167. Move away from the chimney until a point is reached

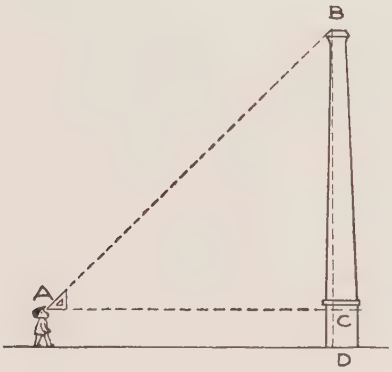


Fig. 167.

where the top of the chimney is sighted along the hypotenuse of the triangle when the triangle is held with the bottom edge horizontal. At the same time, observe the point on the chimney that is level with the lower edge of the triangle. By measuring

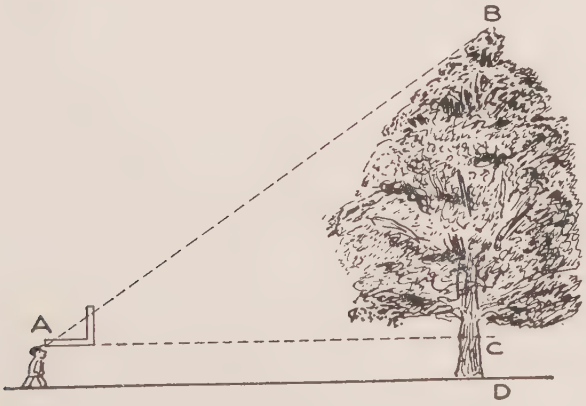


Fig. 168.

the distance of the observer from the chimney and knowing the angle of the triangle, the height can be calculated as before. To this must be added the height from the base to the level of the

bottom edge of the triangle. If a 45° triangle is used, the height of the chimney above the level of the triangle will be just equal to the distance of the observer from the chimney, because the tangent of 45° = 1 and, therefore the distance and height must be equal to each other to give a ratio of 1.

A steel square may also be used, as shown in Fig. 168, placing it so that one leg is horizontal and the other leg vertical, then sighting from a certain point on the horizontal leg and observing where the line of sight crosses the vertical leg. The ratio of these two distances on the square is the tangent of the angle and, therefore, is the same as the ratio of the height of the object to the distance of the observer from the object being measured.

108. Dimensions of V Threads.—The pitch of threads is sometimes stated by giving the number of threads in 1 in. For purposes of calculation it is necessary to use the more exact meaning of pitch; that is, the distance from one thread to the next, as shown by p in Fig. 169.

$$p = \text{pitch} = \frac{1}{\text{Number of threads per inch}}$$

We can calculate the diameter of a bolt at the root of the threads as follows: The angle between the sides of a common V thread

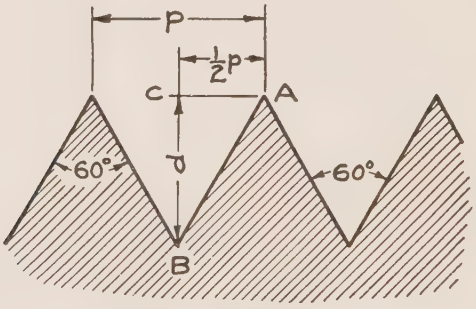


Fig. 169.

is 60°, so a section of a thread will form an equilateral triangle. Hence, in Fig. 169 the angle $ABC = \frac{1}{2}$ of $60° = 30°$

$$\cot ABC = \frac{BC}{AC} = \frac{d}{.5\,p}$$

$$d = .5\,p \times \cot 30°$$

$$d = .5\,p \times 1.732 = .866\,p.$$

The diameter at the root of the threads is equal to the outside diameter minus the depth of thread on each side, or twice the depth of the thread. Hence, letting D = diameter of bolt, d = depth of thread, and D_1 = the diameter at base of the threads,

$$D_1 = D - 2\ d$$
$$D_1 = D - 2\ (.866\ p), \text{ or}$$
$$D_1 = D - 1.732\ p.$$

Since the threads are usually given by the number per inch (N) we can substitute $\frac{1}{N}$ for p. Then

$$D_1 = D - \frac{1.732}{N}$$

109. Dimensions of U. S. Standard Threads.—The United States standard thread is the one generally used on bolts and cap screws. It is similar to the V thread except that the top and bottom are flat, as shown in Fig. 170, instead of being sharp. The angle between the sides is the same (60°).

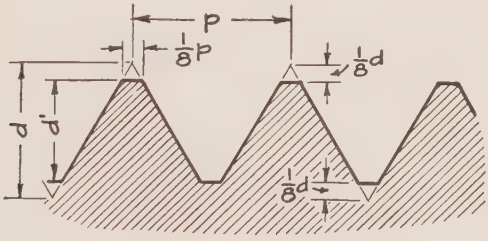


Fig. 170.

If d in the figure represents the depth of the full V, it will be seen that the flattening reduces the depth by $\frac{1}{8}\ d$ at both the top and bottom, leaving the actual depth d', $\frac{3}{4}$ of the height of the full V.

$$d' = \frac{3}{4}\ d = \frac{3}{4}\ (.866\ p), \text{ or } \frac{3}{4}\left(\frac{.866}{N}\right)$$

from which, $\quad d' = .65\ p, \text{ or } \frac{.65}{N}$

Hence, to get the diameter at the root of the threads

$$D_1 = D - 2\ d'$$
$$D_1 = D - 1.3\ p$$
$$D_1 = D - \frac{1.3}{N}$$

Example:

What is the depth of a U. S. standard thread, 8 threads per inch? What is the diameter of a 1-in. bolt at the root of the threads? (A 1-in. bolt has 8 threads per inch.)

$$d' = \frac{.65}{N} = \frac{.65}{8} = .081 + \text{in., } \textit{Answer.}$$

$$D_1 = D - \frac{1.3}{N} = 1 - \frac{1.3}{8} = 1 - .1625 = .8375 \text{ in., } \textit{Answer.}$$

110. Tapers and Taper Turning.—Tapers are usually expressed by the amount of divergence between the opposite sides of the tapering piece. This is stated either by giving the angle between

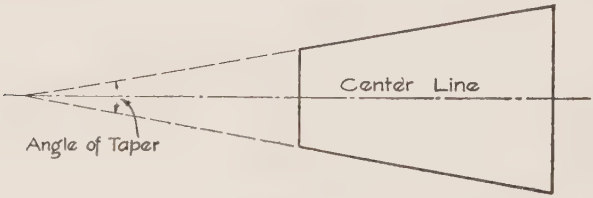


Fig. 171.

these lines, as shown in Fig. 171, or by giving the taper in *inches per foot*, which would be the difference between the widths or diameters at the ends of a piece 1 ft. long. The most common way to express the amount of taper of a piece is to give the taper in inches per foot of length. If a pin 1 ft. in length is 3

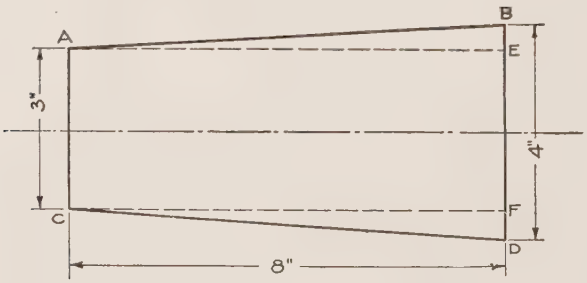


Fig. 172.

in. in diameter at one end and 2 in. in diameter at the other, it is said to taper 1 in. per foot.

Frequently it is necessary to express the taper in degrees. To calculate this change, a table of tangents is necessary. In Fig. 172 the piece tapers 1 in. in a length of 8 in. For a piece 12 in.

long the diameter would change $\frac{1}{8}^2$ of $1=1\frac{1}{2}$ in. Hence this piece tapers $1\frac{1}{2}$ in. per foot. To reduce this taper to degrees of angle between the sides AB and CD, we first draw the lines AE and CF parallel to the center line and making right angles with the end BD. The distances BE and DF are each $\frac{1}{2}$ in.

$$\tan BAE = \frac{BE}{AE} = \frac{0.5}{8} = .0625$$

From this we find in the table that BAE lies between 3° 30′ and 3° 40′ or to be exact is 3° $34\frac{1}{2}$′. The angle FCD, likewise, is 3° $34\frac{1}{2}$′. If one side, AB, slopes 3° $34\frac{1}{2}$′ in one direction from the horizontal while the other side, CD, slopes 3° $34\frac{1}{2}$′ in the other direction, then the angle between the sides, or *the angle of taper*, is $2\times(3°\ 34\frac{1}{2}')=7°\ 9'$.

Sometimes both systems of stating tapers are modified to give only the taper of *one* edge *from the center line*. The piece of Fig. 172 tapers $\frac{3}{4}$ in. per foot from the center line, or it tapers 3° $34\frac{1}{2}$′ from the center line.

In setting a compound rest or a taper attachment graduated to degrees, it should be set to the number of degrees which one side tapers from the center line.

To get a taper in inches per foot when the taper in degrees is given, take half the taper angle. This gives the angle that one side tapers from the center line. Get the tangent of this angle and multiply by 12. This will give the taper from the center line in inches per foot. Now double this and we have the full taper in inches per foot.

There are three possible ways of turning tapers in a lathe. If the lathe has a taper attachment on the back of the bed, that is the best means for turning long tapers. The taper attachment is set at the angle that we want the tool to travel, which is the angle of taper from the center line of the work.

For short tapers and tapers having a considerable angle, the compound tool rest is the best device to use. The tool rest is mounted on a swivel on the carriage and this can be set to feed the tool at any angle desired. The rest is graduated to degrees and it, likewise, must be set to the angle of taper from the center line.

When neither of these devices is at hand, a taper, if not too great, may be turned by setting over the tail stock so that the piece will lie at an angle in the lathe. The tool then travels

straight along the lathe but, since the piece lies at an angle, it will be turned larger at one end than at the other. To calculate the setting for this work, calculate the amount of taper in inches for the entire distance between the lathe centers. The tail stock should be set over one-half of this distance.

It will be noticed that, in all these methods of turning tapers, the taper from the center line is used in setting the lathe. In cutting tapers, the tool must be set level with the lathe centers; otherwise, it will not give the true taper desired.

111. Inverse Functions.—In working from the tangent of an angle to the angle itself, we often have occasion to use expressions like "the angle whose tangent is .5" or "the angle whose cotangent is 3," etc. In writing such expressions in equations and formulas we need some simpler way of stating this. This is done by writing a small $^{-1}$, like an exponent, after *tan* or *cot*. For example "$\tan^{-1}$ .5" means "the angle whose tangent is .5." Likewise, "$\cot^{-1} 3$" means "the angle whose tangent is 3." Such expressions are called *inverse functions*. The small $^{-1}$ is not an exponent and should not be considered as such. This is simply a method of designating an angle by one of its properties instead of by the number of degrees, just the same as we might speak of a certain house up the street as "the house with the slate roof" instead of giving its street number. Remember that "$\tan^{-1}.5$" is not a tangent but is the angle itself.

If we write "$2 \cot^{-1} 1$" it means "2 times the angle whose cotangent is 1." This angle is 45° and twice it is 90°. Hence, $2 \cot^{-1} 1 = 90°$.

These inverse functions are used when we know the value of one of the functions of an angle, like its tangent or cotangent, but either do not have a table handy or do not want to change to degrees. Sometimes we see them used in formulas as, for example, $\tan^{-1} \frac{a}{b}$. Here we cannot reduce the angle to degrees until we know what the values of a and b are for some particular problem.

PROBLEMS

155. At a distance of 300 ft. the angle of elevation of a chimney is 30°, the transit being 4 ft. above the level of the base. Find the height of the chimney.

156. At a distance of 5 miles a mountain has an angle of elevation of 12°. Find the height of the mountain.

157. What is the width of the stream in Fig. 165 if AC is 600 ft. and angle BAC is 42°?

158. What is the root diameter of a $\frac{3}{4}$-in. bolt with 10 U. S. S. threads per inch?

159. In a U. S. S. thread of 4 threads per inch, calculate the width of the flat at the top of the thread.

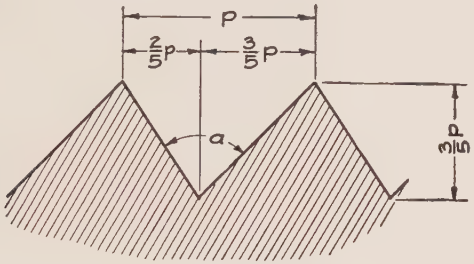


FIG. 173.

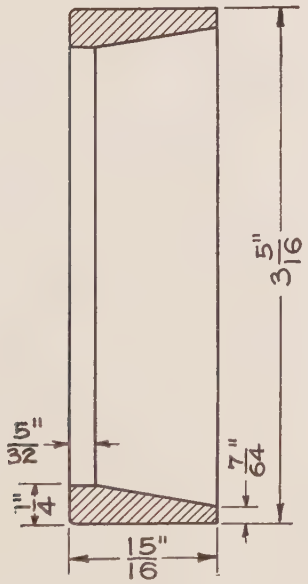


FIG. 174.

160. How many degrees in $3 \times \tan^{-1} .25$?

161. A taper pin $4\frac{1}{2}$ in. long has the large end .492 in. in diameter and the small end .3984 in. What is the taper in inches per foot?

162. A reamer is to be made for standard pins having a taper of $\frac{1}{4}$ in. per foot. In order to be of use for the longest pins of this size, the reamer must be .58 in. in diameter at the point and ,72 in, in diameter at the large end

of the fluted part. How long must the fluted part of the reamer be and what will be the taper from the center line in degrees?

163. In Fig. 173 is shown a special form of screw thread. Such a thread might be used where the pressure was always in one direction, as in the breech block of a cannon. Calculate the angle a to which the nose of a thread tool must be ground to cut this thread.

164. Fig. 174 shows a race for taper roller bearings. From the dimensions given, calculate the angle from the horizontal at which the tool must be fed to bore the taper on the inside of the race.

CHAPTER XVI

THE SINE, COSINE, SECANT, AND COSECANT

112. The Sine.—In addition to the two ratios or functions of angles explained in the two preceding chapters, there are four others; namely, the *sine, cosine, secant, cosecant.* Of these, the most used are the sine and cosine. In any right triangle, the *Sine* of either acute angle is the ratio of the side opposite the angle to the hypotenuse. If AOB in Fig. 175 represents any angle, and we draw PN perpendicular to OB so as to make a right triangle OPN, the sine of the angle at O is the ratio of PN to OP. In writing "the sine of the angle PON" in an equation or formula, it would be abbreviated *sin* PON.

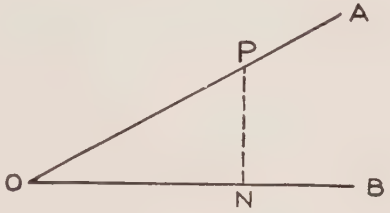


Fig. 175.

Hence, $$\sin PON = \frac{PN}{OP}$$

As will be clearly seen, this ratio is constant for a given angle regardless of how far out from O the line PN is drawn.

The uses of the sine in shop calculations are almost as numerous as the applications of the tangent. In nearly all such problems we have one side of a right triangle given and want to calculate some other side; or else we have two of the sides given and want to get the angle. The relation of these two sides will naturally indicate which function is to be used. A few examples will serve to show some uses of the sine.

Example 1:

Five holes are to be spaced equally about a circle 8 in. in diameter, as shown in Fig. 176. What must be the setting of the dividers to space these holes?

The setting of the dividers will be the length of a straight line AB between any two holes.

If we draw the two radii OA and OB and then draw OC perpendicular to AB, we see that AC and CB are each half of AB; also that, since AOC is a right triangle, the ratio of AC to OA is the sine of the angle AOC.

Now we know that OA is 4 in., being the radius of the circle; and we know that angle AOC is one-half of AOB and that AOB is one-fifth of a circle.

$$AOC = \frac{1}{2} \text{ of } \frac{1}{5} \text{ of } 360° = 36°$$

$$\sin AOC = \frac{AC}{OA}, \text{ from which}$$

$$AC = OA \sin AOC$$
$$AC = 4 \times \sin 36°$$

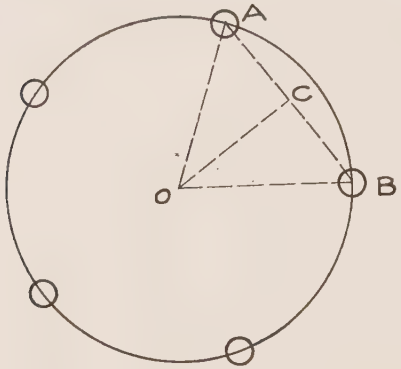


Fig. 176.

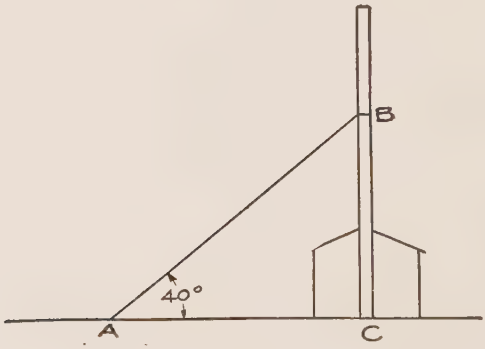


Fig. 177.

From the table in Chapter XIV we find sin 36° to be .5878.

Then, $AC = 4 \times .5878 = 2.3512$ in.

and $AB = 2 \times 2.35 = 4.70$ in., *Answer.*

If the dividers are set to 4.7 in., they will space the 8-in. circle evenly into five parts.

Example 2:

A steel chimney 60 ft. high is to be supported by four guy wires at an angle of 40° with the ground, as shown in Fig. 177; the wires are to be fastened to the chimney two-thirds the way up. How long must each of the guy wires be?

Here we want to find AB when we know BC and the angle BAC.

$$BC = \frac{2}{3} \text{ of } 60 = 40 \text{ ft.} \quad BAC = 40°$$

$$\frac{BC}{AB} = \sin BAC = \sin 40°$$

Then, $BC = AB \times \sin 40°$

or $$AB = \frac{BC}{\sin 40°} = \frac{40}{.6429} = 62.22 \text{ ft., } \textit{Answer.}$$

Each guy wire must be 62.22 ft. long plus, of course, an additional allowance for fastening the ends.

113. The Cosine.—In any right triangle, the *Cosine* of either acute angle is the ratio of the adjacent side to the hypotenuse. In Fig. 175, the cosine of the angle PON is the ratio of ON to OP. In writing "the cosine of angle PON" in an equation or formula it would be abbreviated *cos PON*.

Hence, $$\cos PON = \frac{ON}{OP}$$

The cosine would naturally come into use in shop calculations which depend on the relation between an angle and its adjacent side and the hypotenuse in a right triangle.

Example:

An octagon is to be milled out of 2-in. round stock. What will be the short diameter of the largest octagon that can be made from it?

Referring to Fig. 178, it will be seen that if we draw the radii AO and BO from two adjacent corners A and B, the angle AOB will be $\frac{1}{8}$ of $360° = 45°$.

The dimension to be found is DC, and this is just twice OC. In the right triangle AOC, OC is the side adjacent to the angle AOC and OA is the hypotenuse. Hence $OC \div OA$ is the cosine of AOC.

$$\frac{OC}{AC} = \cos AOC = \cos 22\tfrac{1}{2}°$$

$$OC = AC \times \cos 22\tfrac{1}{2}°$$

$$= 1 \times .9239 = .9239$$

$$DC = 2 \quad OC = 2 \times .9239 = 1.8476 \text{ in., } \textit{Answer.}$$

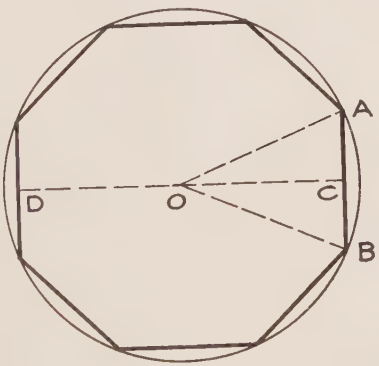


FIG. 178.

Note.—The width of one face of the octagon would be found by the use of the sine.

$$AB = 2\ AC = 2\ OA \sin 22\tfrac{1}{2}° = .7654 \text{ in.}$$

The cosine derives its name from the fact that it is the sine of the complementary angle, just as the cotangent of an angle is the tangent of the complementary angle. In Fig. 175, the

cosine of angle PON is $ON \div OP$, and this is readily seen to be the sine of the complementary angle OPN.

Hence, the sine of an angle = cosine (90° − the angle)
or the cosine of an angle = sine (90° − the angle)

This explains why the same column in the tables will give sines when reading from the angles at the left of the page and using the heading at the top, and will give cosines when reading from the angles at the right side and using the column heading at the bottom of the page.

114. The Secant.—The secant of an angle is the reciprocal of the cosine; that is, it is the reverse ratio of the cosine. In a right triangle, the *Secant* of an angle is the ratio of the hypotenuse to the side adjacent to the angle. The chief use of the secant is to enable one to multiply where it would be necessary to divide if the cosine were used. The secant is abbreviated *sec.*

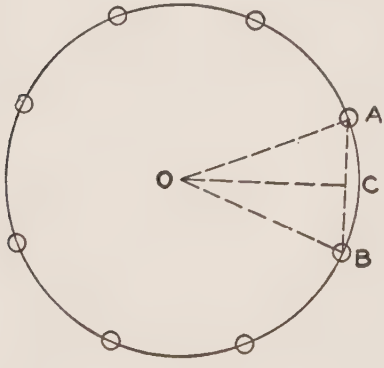


FIG. 179.

Example:

What size of round bar is needed to mill an octagon having a short diameter of $1\frac{1}{2}$ in.?

Referring to Fig. 178, it will be seen that OC in this case is $\frac{1}{2}$ of $1\frac{1}{2} = \frac{3}{4}$ in. The dimension wanted, or the diameter of the round bar, is $2 \times OA$.

$$OA = OC \div \cos 22\tfrac{1}{2}^\circ = OC \times \sec 22\tfrac{1}{2}^\circ$$
$$= .75 \times 1.0818 = .8113$$
$$2\,OA = 2 \times .8113 = 1.6226 \text{ in., } \textit{Answer.}$$

The stock needed would therefore be practically $1\frac{5}{8}$ in. in diameter.

115. The Cosecant.—The *Cosecant* is the reciprocal of the sine; that is, it is the reverse ratio, being the ratio of the hypotenuse to the side opposite the angle. The chief use of the cosecant is in avoiding the necessity of dividing by the sine. The abbreviation for the cosecant is *csc.*

$$\text{csc of an angle} = \frac{1}{\text{sin of the angle}}$$

Example:

A draftsman wants to space 8 bolts equally around a circle so that it will be approximately 4 in. from one bolt to the next. What should be the diameter of the circle?

Referring to Fig. 179:

$$AB = 4 \text{ in., or } AC = 2 \text{ in.}$$
$$OA = AC \div \sin 22\tfrac{1}{2}^\circ = AC \times \csc 22\tfrac{1}{2}^\circ$$
$$= 2 \times 2.613 = 5.226 \text{ in., } \textit{Answer.}$$

Hence, the bolts would be on a circle of about $5\frac{1}{4}$-in. radius.

The relation of the six trigonometric ratios can be seen in the following diagram.

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot = \frac{\text{adjacent}}{\text{opposite}} \qquad \sec = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \csc = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sin = \frac{1}{\csc} \qquad \csc = \frac{1}{\sin}$$

$$\cos = \frac{1}{\sec} \qquad \sec = \frac{1}{\cos}$$

$$\tan = \frac{1}{\cot} \qquad \cot = \frac{1}{\tan}$$

116. Laying out Angles.—Angles can be laid out from the sine, cosine, secant, or cosecant just as readily as from the tangent or cotangent, and in some cases it is more convenient, if the value of some function is an even figure, to use that function.

Example 1:

Construct the angle whose sine is .5, or $\sin^{-1} .5$.

First draw the base line RX of any length, Fig. 180, and erect a perpendicular NP of any length. The hypotenuse must be twice NP in length in order to give an angle whose sine is .5. About P as center, swing an

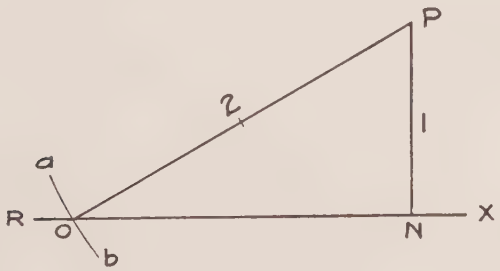


Fig. 180.

arc AB with a radius $OP = 2 \times NP$. From the intersection O of this arc with RX draw the line OP, thus completing the triangle.

OP is by construction twice NP.

Hence, $\sin NOP = \frac{1}{2} = .5$

and angle $NOP = \sin^{-1} .5$

Example 2:

Construct $\cos^{-1} .7$.

$\text{Cos}^{-1} .7 = \cos^{-1} \frac{7}{10}$. Hence, the ratio of the adjacent side to the hypotenuse must be 7 : 10.

On any line RX, Fig. 181, lay off ON equal to 7 units. At N erect the perpendicular NM. About O swing an arc ab with radius $OP = 10$ units and from the intersection P with MN draw OP.

$$\cos NOP = \frac{ON}{OP} = \frac{7}{10} = .7;$$

and, therefore, angle $NOP = \cos^{-1} .7$, or, in words, "The angle NOP equals an angle whose cosine is .7."

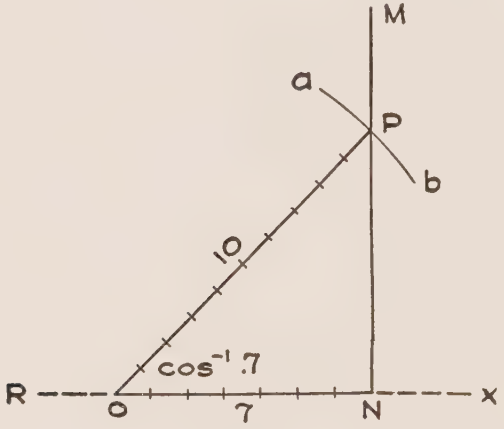


Fig. 181.

117. Use of the Tables.—In the tables at the end of Chapter XIV will be found the values of all the functions or ratios, sin, cos, tan, cot, sec, and csc for all angles from 0° to 90°. The same instructions apply to the other functions as were given for the tangent and cotangent. Always remember in looking for functions of angles over 45° to refer to the bottom of the page to find the column in which to look for the function desired, and refer to the right-hand side of the page to find the angle.

118. To Find Intermediate Values.—As is well known, degrees are subdivided into minutes, there being 60 minutes in a degree; and minutes are subdivided into seconds, there being 60 seconds in a minute. The mark for designating minutes is the same as that commonly used by draftsmen to indicate feet (′) and the mark for seconds is that used for inches (″).

Thus, 36° 43′ 21″ is read 36 degrees, 43 minutes, and 21 seconds. If we want to reduce this to degrees and minutes,

$$21'' = \frac{21}{60} \text{ min.} = .35'$$

$$\text{Hence, } 36° 43' 21'' = 36° 43.35'$$

Now suppose we want the sine of this angle. Having reduced the seconds to a fraction of a minute, we can find from the table the sines of 36° 40′ and 36° 50′.

$$\begin{array}{r} \sin 36° 50' = .5995 \\ \sin 36° 40' = .5972 \\ \hline \text{Difference for } 10' = .0023 \end{array}$$

For 36° 43.35′ we would add $\frac{3.35}{10}$ of this difference to sin 36° 40′ to get sin 36° 43.35′.

$$.335 \times .0023 = .0008 -$$

$$\text{Hence, } \sin 36° 43' 21'' = .5972 + .0008 = .5980$$

It will be noticed that some functions decrease instead of increasing as the angle increases and, in such cases, the correction must be *subtracted* instead of added. All of the *co-functions* (functions whose names begin with *co*) *decrease*. As an example, let us find cos 49° 46′. From the table we find cos 49° 40′ and cos 49° 50′

$$\begin{array}{r} \cos 49° 40' = .6472 \\ \cos 49° 50' = .6450 \\ \hline \text{Difference for } 10' = .0022 \end{array}$$

Since cos 49° 50′ is *less* than cos 49° 40′ we would *substract* $\frac{6}{10}$ of this difference from cos 49° 40′ to get cos 49° 46′.

$$.6 \times .0022 = .0013 +$$

$$\text{Hence, } \cos 49° 46' = .6472 - .0013 = .6459$$

For most practical purposes it is not necessary to bother with the seconds, since it is hardly possible to measure angles on a small scale so accurately. In laying out an angle on a 10-in. base line by means of the tangent, an error of .01 in. means an error of about 5′, so it would be useless to figure closer than 1 minute. On a protractor having an 8 in. circle, an error of .01 in. along the circle means an error of about $8\frac{1}{2}'$; so here, also, an accuracy of 1′ is as much as can possibly be expected.

PROBLEMS

165. Find the values of the following functions from the tables:

sin 14° 28′	cos 70° 52′
tan 63° 26′	sec 48° 12′
csc 24° 35′	cot 36° 1′

166. Find from the tables the values in degrees and minutes of the following angles:

$\tan^{-1} 2.$	$\csc^{-1} 1.25$
$\sin^{-1} 0.3$	$\sec^{-1} 1.05$
$\cos^{-1} 0.7$	$\cot^{-1} 0.875$

167. Construct the following angles by means of their functions:

$\sin^{-1} 0.2$	$\tan^{-1} 2.0$
$\cos^{-1} 0.6$	14° 28′

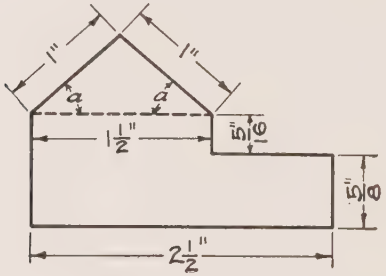


Fig. 182.

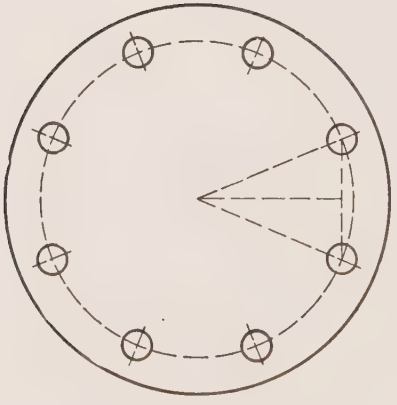


Fig. 183.

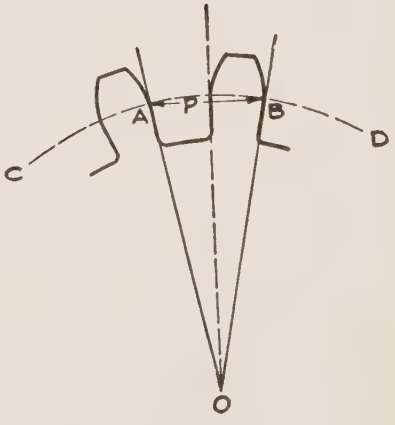


Fig. 184.

168. What is the length of wire needed to support a steel chimney 60 ft. high, there being 4 guys, fastened $\frac{2}{3}$ the way up the stack and making an angle of 35° with the ground? Allow 20 ft. extra for fastening the ends.

169. A steel block is to be finished to the dimension shown in Fig. 182.

Calculate the angle of inclination a at which to plane the two sloping surfaces.

170. The cylinder head of Fig. 183 is to be laid out with 8 holes equally spaced on a circle of 10 in. diameter. Calculate the distance between centers of adjacent holes so that they can be laid out with the dividers.

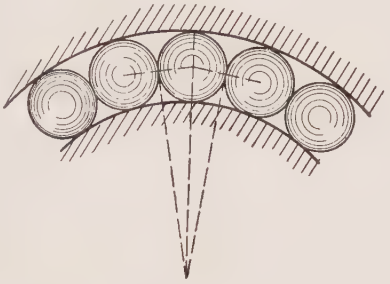


FIG. 185.

171. The straight distance p from A to B in Fig 184 is known as the *chordal pitch* of the gear because it is measured by a straight line or *chord* of the pitch circle CD, instead of along the curve of the circle. If the radius AO is 2 in. and the gear has 16 teeth, what would be the chordal pitch p?

172. If 20 half-inch balls are to be placed in a circle as shown in Fig. 185, calculate the diameters of the inner and outer ball races.

CHAPTER XVII

SCREW THREADS AND SPIRALS

In Chapter XV the shapes and dimensions of the V thread and of the U. S. standard thread were explained. There are a great many other threads that every shop man should know about, some of them being in quite common use for certain purposes.

119. Acme Screw Threads.—Fig. 186 shows the form of this thread. Its principal use is in machine tool manufacture, where it is largely used for lead screws and other service which requires that the screw must transmit power. The working depth, or the depth in which there is contact, equals one-half the pitch. The grooves are recessed .01 in. deeper on all sizes

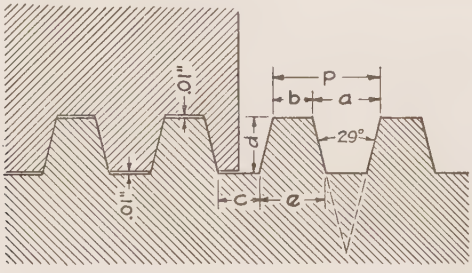


Fig. 186.

for clearance, thus leaving a small space for oil and dirt to collect in. Because of this clearance, the width of the bottom of the space is slightly narrower than the width of the top of the thread. The following formulas give the dimensions in terms of the pitch:

p = pitch	$= \frac{1}{\text{No. of threads per inch}}$
d = depth of thread	$= .5\ p + .01$ in.
a = space at top	$= .6293\ p$
b = thickness at top	$= .3707\ p$
c = space at bottom	$= .3707\ p - .0052$ in.
e = thickness at bottom	$= .6293\ p + .0052$ in.

120. Brown and Sharpe 29° Worm Threads.—This thread

shown in Fig. 187, is similar to the Acme thread but is deeper, It is used for similar purposes but finds its chief use in worms for worm gearing and it is from this use that it derives its form, which is the same as that of the tooth on a rack of the involute

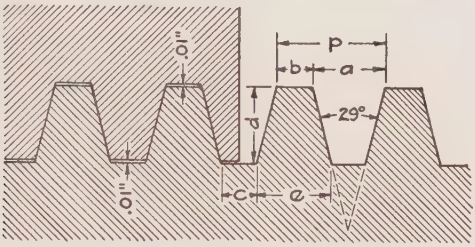


Fig. 187.

system of gearing. The following formulas give the dimensions in terms of the pitch.

p = pitch	$= \frac{1}{\text{No. of threads per inch}}$.
d = depth of thread	$= .6866\ p$
d' = working depth	$= .6366\ p$
a = space at top	$= .665\ p$
b = thickness at top	$= .335\ p$
c = space at bottom	$= .310\ p$
e = thickness at bottom	$= .690\ p$

121. Briggs Standard Pipe Threads.—This thread has an angle of 60° similar to the V and U. S. standard threads. The

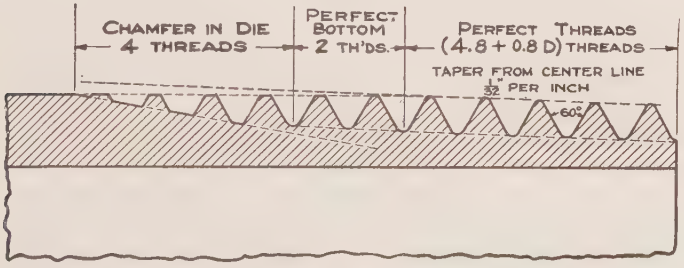


Fig. 188.

top and root are slightly rounded, so that the depth of thread is but .8 p. The depth of a full V thread of 60° angle is .866 p, so it will be seen that the Briggs thread is rounded but a very little and in the small pitches the difference from the V thread is almost imperceptible.

In threading a pipe, the threads are cut on a taper of $\frac{3}{4}$ in. per foot, or $\frac{1}{16}$ in. per inch. The taper on each side of the center line is, consequently, $\frac{1}{32}$ in. per inch of length. Fig. 188 shows the shape of thread as well as the taper of the ends of the pipe.

122. Whitworth Standard Screw Threads.—This is an English thread and is seldom found in this country except on machinery imported from England. As shown in Fig. 189, the angle between the sides of the threads is 55°, instead of 60° as in the American V thread and in the U. S. standard. The point and root are rounded off on an arc of a circle so that the full height of the *V*

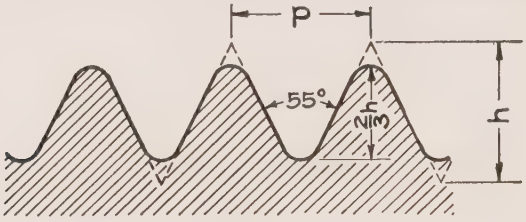


Fig. 189.

is reduced $\frac{1}{6}$ at each point. This leaves the depth of the thread $\frac{2}{3}$ of that of a full V thread of 55°.

123. British Association Screw Threads.—This thread is similar in appearance to the Whitworth thread but is even steeper, the angle between the sides being $47\frac{1}{2}°$. It is rounded at the point and root so as to give a depth equal to .6 of the pitch.

124. French (or Metric) Standard Screw Threads.—The form of this thread is the same as that of the U. S. standard thread. It will be understood that bolt diameters in this system are given in millimeters and the pitch is also given in millimeters. When we compare the two systems, we find that for bolts of nearly the same size the pitch is smaller in the French system than in the U. S. system. However, the threads are identical in that they both have the 60° angles and are both flattened at the top and root in the same proportions.

125. International Standard Screw Threads.—The thread is in form the same as the French and, therefore, also as the U. S. standard. The only difference from the French system lies in the fact that for a few sizes of bolts the pitch is different.

126. Buttress Threads.—The buttress thread is a thread sometimes used where the pressure is always in one direction. There is no fixed standard, although the proportions shown in

Fig. 190 are the most common. One face is vertical and the other is inclined at an angle of 45°. The top and bottom are left flat as in the U. S. standard thread so as to reduce the depth to $\frac{3}{4}$ of what it would be if the thread were cut to a sharp point at the top and bottom.

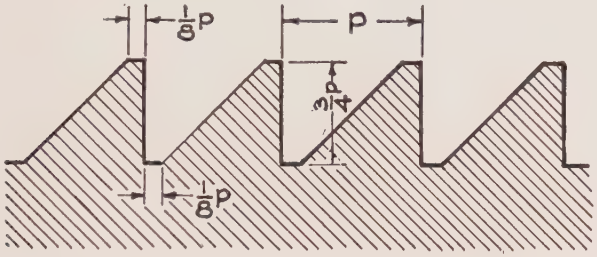


Fig. 190.

127. Square Threads.—The chief use of the square thread-shown in Fig. 191, is in jack screws and similar screws sub, jected to very heavy loads. The working depth is equal to one-half the pitch, but the threads are usually cut .01 in. deeper, thus giving a clearance of this amount at the top and bottom. The depth is therefore obtained by the formula

$$d = .5\ p + .01 \text{ in.}$$

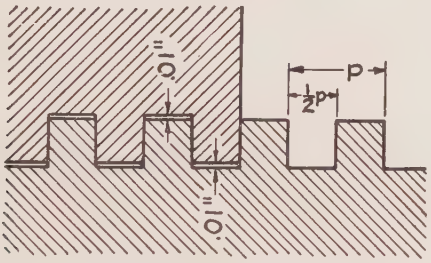


Fig. 191.

The groove is also cut a slight amount wider than the thread, to allow a free sliding fit between the threads. There is no definite standard of pitches for square threads although a common rule is to make the pitch just twice as great as that for a U. S. standard bolt of the same diameter.

128. Pitch and Lead—Helix Angles.—If we take a triangle and wrap it around a cylinder, in the manner shown in Fig. 192, we will have a *helix* formed by the hypotenuse of the triangle.

This helix is exactly like a screw thread in the manner in which it advances along the cylinder, and in fact every screw thread is a helix.

The distance which the helix or thread advances along the cylinder in one revolution is called the *lead*. It will be seen that this is the same as the pitch of single threads. This is the angle at which the thread is wound and is called *the helix angle*, *the angle of advance*, or often simply *the thread angle*. The tangent of this angle equals the lead divided by the circumference, or

$$\tan a = \frac{\text{lead}}{\text{circumference}}$$

This angle will vary at different depths. The lead is the same at all points on a thread, but the circumference is greater

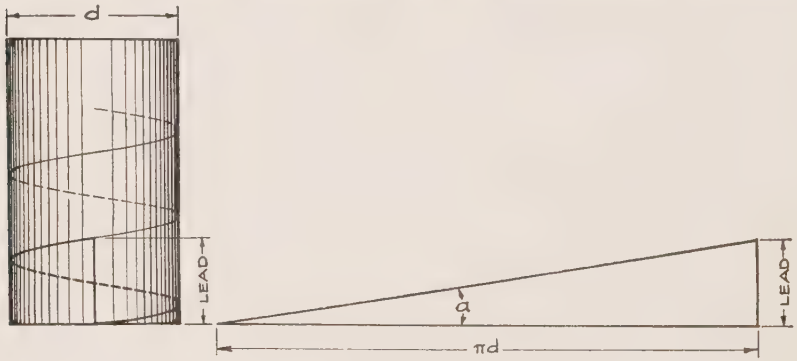


Fig. 192.

at the top of the thread than at the root. Consequently, we generally use the average circumference, or 3.1416 × the average diameter.

Average diameter = Outside diameter − 2 × ($\frac{1}{2}$ depth of thread)
= Outside diameter − Depth of thread.

Another way of expressing this is to get half the sum of the diameter at the top of the threads and the diameter at the base of the threads.

If D = outside diameter or diameter at top of threads and d = diameter at base of threads, then the average diameter = $\frac{D+d}{2}$; and

$$\tan a = \frac{\text{lead}}{\pi \times \frac{1}{2}(D+d)}$$

In cutting a thread, the tool advances in one revolution a distance equal to the lead, while the work turns through a distance equal to the circumference. Therefore, the tool should be ground or set at this angle *a*. On lathe tools this is sometimes called the *side angle*, it being the angle at which the nose of the tool slopes from the vertical when viewed from the cutting end. In many shops, a number of thread tools are kept in stock, each ground to a different helix angle. Each tool is used within a certain range of angles.

129. Multiple Threads.—Fig. 193 shows what is meant by a

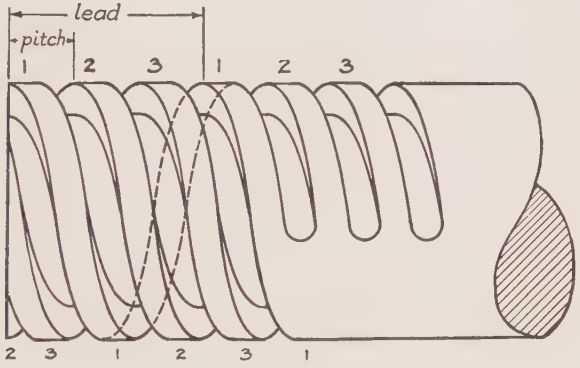


Fig. 193.

multiple thread. In this case we have a triple thread, consisting of three distinct threads cut in the bar. The pitch is still the distance from one thread to the next; but the lead is the distance that one thread advances along the screw in one revolution, as shown in Fig. 193. In a triple thread the lead is three times the pitch. Multiple threads are used where it is desired to give the nut a considerable advance in each revolution and yet use a small pitch thread so as not to weaken the screw by cutting a large and deep thread. For the same lead, a triple thread would have threads one-third as large as a single thread; and a double thread would have threads one-half as large as a single thread of the same lead. Change gears for cutting multiple threads must be figured according to the *lead* of the thread to be cut rather than the pitch. Each thread is cut separately.

130. Spiral Gears.—When the teeth of a gear are cut as shown in Fig. 194, in a helical or screw-like manner instead of straight across, the gear is called a *spiral gear*. Strictly speaking, a true spiral is a curve like that of a watch spring, but usage

has given that name to gears having a screw or helical tooth. A spiral gear is in reality a multiple thread screw, and the thread angle can be obtained in the same way by dividing the lead by the circumference. However, because of the methods of cutting them in a universal milling machine, the angle of a spiral or the spiral angle is usually given as the angle between the tooth and a line drawn parallel to the axis of the gear. This angle is 90° minus the thread angle. The difference between these two angles is shown in Fig. 195. The angles are complementary. Hence, the tangent of the thread angle is the cotangent of the spiral angle and the cotangent of the thread angle is the tangent of the spiral angle.

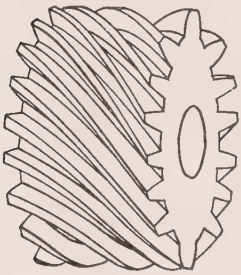


Fig. 194.

The tangent of the spiral angle is obtained by dividing the circumference by the lead. In calculating spiral angles the circumference at the pitch line, or the middle of the working depth of the tooth, is used.

In cutting a spiral gear on a milling machine, the table must be swung from its usual position through an angle equal to the spiral angle. The arbor carrying the gear blank must then be

Fig. 195.

geared so as to revolve once while the table is fed a distance equal to the lead of the spiral.

131. Relations between Trigonometric Functions.—If one function of an angle is known, it is possible to calculate all the others from it. The relations between the functions are easily

seen and can be all derived from the definitions of the functions and from the law of right triangles. In addition to the relations given in Chapter XVI, we have the following relations:

Let A in Fig. 196 represent any angle, and let a, b, and c be sides of a right triangle having A for one of its acute angles.

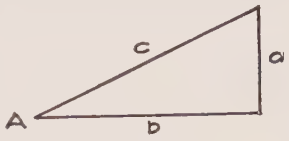


FIG. 196.

We know that $a^2+b^2=c^2$

Dividing by c^2 we get $\frac{a^2}{c^2}+\frac{b^2}{c^2}=1$

But $\frac{a}{c}=\sin A$, and $\frac{b}{c}=\cos A$; hence

$$(\sin A)^2+(\cos A)^2=1 \tag{1}$$

From this we get

$$\sin A=\sqrt{1-(\cos A)^2} \tag{2}$$

$$\cos A=\sqrt{1-(\sin A)^2} \tag{3}$$

$$\frac{\sin A}{\cos A}=\frac{a}{c}\div\frac{b}{c}=\frac{a}{b}=\tan A\text{; or}$$

$$\tan A=\frac{\sin A}{\cos A} \tag{4}$$

By combining (4) with (2) and (3) we get

$$\tan A=\frac{\sin A}{\sqrt{1-(\sin A)^2}} \tag{5}$$

$$\cot A=\frac{\cos A}{\sqrt{1-(\cos A)^2}} \tag{6}$$

By combining (4) and (1) we can get

$$(\cos A)^2=\frac{1}{1+(\tan A)^2} \tag{7}$$

$$(\sin A)^2=\frac{(\tan A)^2}{1+(\tan A)^2}=\frac{1}{1+(\cot A)^2} \tag{8}$$

PROBLEMS

173. Calculate the diameter at the root of the thread for a $1\frac{3}{4}$-in. bolt with Whitworth threads. There are 5 threads per inch.

174. Work out a formula for the diameter at the root of the threads for any screw having the Whitworth standard threads. Follow the method used in Chapter XV for the V and U. S. S. threads.

175. A 2-in. screw is to have a double thread of the Acme standard shape with 2 threads per inch. How far will a nut advance along the screw in one revolution and what is the helix angle of the screw?

176. Starting with the $\sin 30°=0.5$, calculate, by means of the equations in Chapters XVI and XVII, the other functions—cosine, tangent, cotangent, cosecant, and secant.

177. If we have the functions of a certain angle, we can find the functions of *half* that angle by using the following formula together with those in Chapters XVI and XVII.

$$(\sin \tfrac{1}{2} A)^2 = \tfrac{1}{2}(1 - \cos A)$$

Starting with $\cos 30° = .866$ find the sine, cosine, and tangent of 15°.

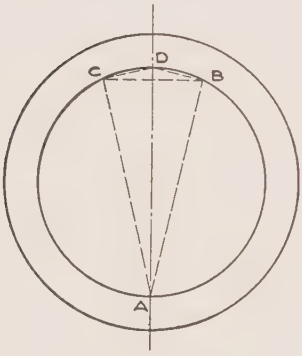


Fig. 197.

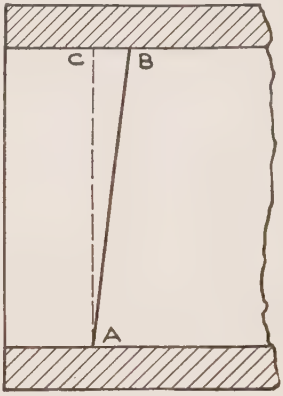


Fig. 198.

178. Calculate the helix angle for the thread of a $1\frac{1}{2}$-in. U. S. S. bolt with 6 threads per inch.

179. What should be the width at the nose of a thread tool to cut an Acme thread with 6 threads per inch?

180. Allowances for running fits are sometimes made by boring the hole large enough so that an inside caliper or an end measuring rod will have a certain amount of play as shown in Fig. 197, AC and AB being the extreme positions of the measuring rod and CB being the amount of side play. To find the true diameter AD, the following method can be followed: knowing AB and CB, the angle DAB or DAC on either side of the center-line can be calculated. Having found this angle and knowing AB, calculate the diameter AD. The angle ABD is a right angle because B lies on the circumference of a circle and A and D are at opposite ends of a diameter.

If a 6-in. standard end measuring rod is given a side play of $\frac{1}{2}$ in. in a hole, what is the actual diameter of the hole?

Fig. 199.

181. Fig. 198 shows a method of allowing for force or shrink fits, the hole being bored small so that the calipers or gage will stand in a position AB, a certain distance BC from the vertical. If the calipers are given an inclination, BC, of $\frac{1}{16}$ in. for each inch of diameter, what shrinkage allowance per inch of diameter does this provide?

182. The grooves in the twist drill of Fig. 199 make one complete turn in a length of 7 diameters. Calculate the spiral angle of the grooves.

CHAPTER XVIII

SOLUTION OF TRIANGLES

132. Functions of Angles Greater than 90°.—Up to this point we have dealt only with the functions of angles less than 90° and have confined our study of trigonometry to problems involving only right triangles. We often have to make calculations concerning oblique triangles, in which one angle is greater than 90°.

In getting the functions of any angle, we drop a perpendicular from a point along one side of the angle to the other side, as PN in Fig. 200. Then the functions of this angle AOB are obtained from the various ratios of the sides PN, ON, and OP. Thus,

$$\sin AOB = \frac{NP}{OP};\ \cos AOB = \frac{ON}{OP};\ \tan AOB = \frac{NP}{ON},\ \text{etc.}$$

If an angle is greater than 90°, we cannot draw this perpendicular PN, Fig. 201, except by extending AO on the opposite side of O and drawing PN perpendicular to this extension ON.

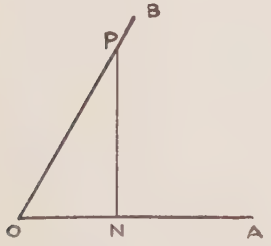


Fig. 200.

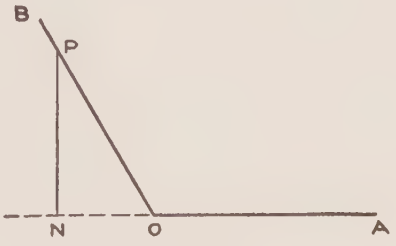


Fig. 201.

We can now use these lines PN, ON, and OP to get the functions of this oblique angle AOB. The functions of this angle would then be the same in value as those of the smaller supplementary angle PON, but with this difference: *The distance ON, being measured in the opposite direction from OA, is considered as being*

a negative quantity. OP and PN are still considered as positive quantities. Our functions will be the same ratios as before:

$$\sin AOB = \frac{PN}{OP} \qquad \cos AOB = \frac{ON}{OP}$$

$$\tan AOB = \frac{PN}{ON} \qquad \cot AOB = \frac{ON}{PN}$$

$$\sec AOB = \frac{OP}{ON} \qquad \csc AOB = \frac{OP}{PN}$$

but it will be seen that, since ON is a negative quantity, all the functions involving ON—namely, the cosine, tangent, cotangent, and secant—will be negative in value. Otherwise, the functions of AOB will be exactly the same in value as those of the supplementary angle PON.

$$\sin AOB = \sin PON \qquad \cos AOB = -\cos PON$$
$$\tan AOB = -\tan PON \qquad \cot AOB = -\cot PON$$
$$\sec AOB = -\sec PON \qquad \csc AOB = \csc PON$$

For example, suppose AOB is an angle of 120°. Then the supplementary angle PON will be $180-120=60°$.

$$\sin 120° = \sin 60° = .8660 \qquad \cos 120° = -\cos 60° = -.5000$$
$$\tan 120° = -\tan 60° = -1.732 \qquad \cot 120° = -\cot 60° = -.5774$$
$$\sec 120° = -\sec 60° = -2. \qquad \csc 120° = \csc 60° = 1.155$$

Thus, it will be seen that to get from a table the functions of an angle between 90° and 180° we subtract the angle from 180° and find the functions of this supplementary angle; at the same time we must remember to place the minus sign (−) before those functions involving the adjacent side ON, namely, the cosine, tangent, cotangent, and secant.

133. Functions of 0°.—In a small angle such as shown in Fig. 202, the perpendicular PN drawn from one side to the other is relatively small, while ON and OP are nearly equal. In an angle of 0°, OP and ON would coincide and be equal, while PN would be zero.

Since PN is zero, it follows that the sine and tangent of 0° are both zero in value.

Since ON and OP coincide and are equal, the cosine of 0° is 1; likewise the secant of 0° is 1.

The values of the cotangent and cosecant are more difficult to see. As the angle PON of Fig. 202 is made smaller and smaller,

the length of line PN becomes very small in comparison with OP and ON. Consequently, the values of the cotangent $\left(=\frac{ON}{PN}\right)$ and the cosecant $\left(=\frac{OP}{PN}\right)$ become very large, as can be seen from the tables. When the angle becomes zero, PN also becomes zero. Therefore, the cotangent and cosecant become infinitely large and their values are said to be *infinity*. The mathematical sign of infinity is ∞, meaning a number whose value is greater than any number which we can imagine. In these two functions we have a fraction or ratio having zero in the denominator; and any quantity divided by zero gives infinity for the quotient.

Fig. 202.

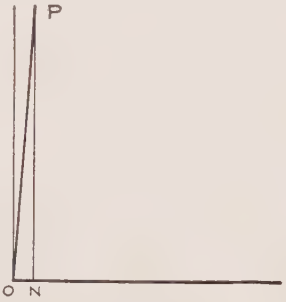


Fig. 203.

Then we have the following values for the functions of an angle of 0°:

$\sin 0° = 0$	$\cos 0° = 1$	$\tan 0° = 0$
$\csc 0° = \infty$	$\sec 0° = 1$	$\cot 0° = \infty$

134. Functions of 90°.—Fig. 203 shows an angle PON approaching 90°. The length of ON is very small while OP and ON are nearly equal. When the angle becomes 90°, ON will be zero and OP and ON will be equal. Consequently, we have the following values of the functions of 90°:

$\sin 90° = 1$	$\cos 90° = 0$	$\tan 90° = \infty$
$\csc 90° = 1$	$\sec 90° = \infty$	$\cot 90° = 0$

Fig. 204.

135. Functions of 180°.—Fig. 204 shows an angle AOP approaching 180°. The length of PN is very small, while OP and ON are nearly equal in value, except that ON is negative, being measured to the left from O. When the angle is made exactly

180°, PN becomes zero and PO and ON are equal in value, but ON is negative. Consequently, we have the following values of the functions of 180°:

$$\sin 180° = 0 \qquad \cos 180° = -1 \qquad \tan 180° = 0$$
$$\csc 180° = \infty \qquad \sec 180° = -1 \qquad \cot 180° = -\infty$$

136. Solution of Triangles.—In the preceding chapters, we have confined our applications of trigonometry to problems involving right triangles. We will now study the principles by which it can be applied to solving triangles of any shape. By *solving a triangle* we mean that we have some of the sides and angles given and proceed to calculate the rest. There are two laws or formulas that are needed for such work and these are called the *Law of Sines* and the *Law of Cosines.*

We will first take up the use of the law of cosines because it is the extension of our law of right triangles so as to be applicable to any triangle.

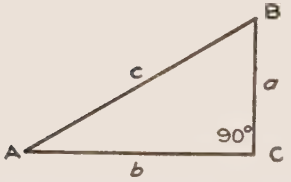


Fig. 205.

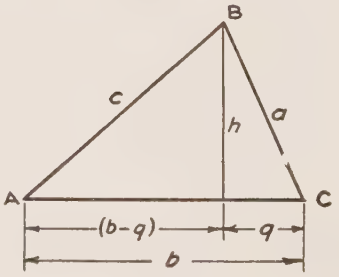


Fig. 206.

137. Law of Cosines.—We have often used the law that "the square of the hypotenuse of any right triangle is equal to the sum of the squares of the other two sides"; or, referring to Fig. 205,

$$c^2 = a^2 + b^2$$

By a slight change in this formula we can make it applicable to any triangle.

Let Fig. 206 represent any triangle whatever, in which we know the sides a and b and the angle C and want to get the length of the side c. As a first step toward developing the law, let us draw h perpendicular to b and let the two parts into which b is cut by h be designated by q and $(b-q)$. Then we will have two right triangles in which

$$c^2=h^2+(b-q)^2 \tag{1}$$
$$\text{or } c^2=h^2+b^2-2\,bq+q^2 \tag{2}$$
$$\text{and } \quad h^2=a^2-q^2 \tag{3}$$

If we put the value of h^2 in (3) into the second equation in the place of h^2 we will have:

$$c^2=a^2-q^2+b^2-2\,bq+q^2$$
$$c^2=a^2+b^2-2\,bq \tag{4}$$

We see some resemblance here to the law of right triangles but we still have q in the equation and that is not one of the things that were known about the triangle. But we can see that $q=a\cos C$ and, if we put this value for q in the previous equation (4), we get the Law of Cosines:

$$c^2=a^2+b^2-2\,ab\cos C \tag{5}$$

In the case that we just considered, the angle C was less than 90°. We will now see how this law would operate in case C should be greater than 90°, as in Fig. 207. Going through a series of equations similar to the previous case we have:

$$c^2=h^2+(b+q)^2 \tag{6}$$
$$\text{or, } c^2=h^2+b^2+2\,bq+q^2 \tag{7}$$
$$\text{and, } h^2=a^2-q^2 \tag{8}$$

Inserting for h^2 in (7) its value as found in (8) we have:

$$c^2=a^2-q^2+b^2+2\,bq+q^2$$
$$c^2=a^2+b^2+2\,bq \tag{9}$$

In this case, $q=a\cos BCX$ and, therefore:

$$c^2=a^2+b^2+2\,ab\cos BCX \tag{10}$$

Here, instead of coming out with the angle C, as in (5), we have BCX which is $(180-C)$. We also have the sign of this last term + instead of −. According to the explanation in Art. 132 of the functions of angles between 90° and 180°, the cos C is the same as cos BCX except that the sign is different, that is

$$\cos C=-\cos BCX$$
$$\text{or} -\cos C=\cos BCX$$

and, by putting this value of cos BCX in our equation (10) we have

$$c^2=a^2+b^2+2\,ab\,(-\cos C)$$
$$c^2=a^2+b^2-2\,ab\cos C \tag{11}$$

This makes the law the same for either case. In using this law of cosines, you must remember that in case C is greater than 90° its cosine will be minus and, therefore, the last term will be minus a minus quantity, which makes the term plus. You can easily follow this if you will just look at the triangles and notice that if C is less than 90° then c^2 will be *less* than a^2+b^2; and when C is greater than 90° then c^2 will be *greater* than a^2+b^2.

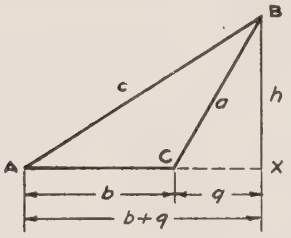


Fig. 207.

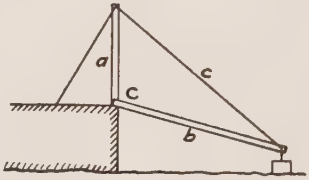


Fig. 208.

In the same way, we can derive rules for a^2 or b^2 in terms of the other sides.

$$a^2=b^2+c^2-2\,bc\,\cos A$$
$$b^2=a^2+c^2-2\,ac\,\cos B$$
$$c^2=a^2+b^2-2\,ab\,\cos C$$

Notice particularly that the given angle must be the angle included between the two given sides.

Example:

In the crane of Fig. 208 the height of the mast a is 30 ft. and the length of the boom b is 40 ft. What must be the length of a single cable at c in order that the boom may be lowered to make an angle C of 130°?

$c^2=a^2+b^2-2\,ab\,\cos C$
$\cos C=\cos 130°=-\cos(180-130)=-\cos 50°=-.6428$
$c^2=30^2+40^2-2\times30\times40\times(-.6428)$
$=900+1600+1542.7$
$c^2=4042.7$
$c=\sqrt{4042.7}=63.58+$ ft., *Answer.*

The law of cosines can also be transposed into a formula by which we can get any angle of a triangle when the three sides are given. Taking the formula $c^2=a^2+b^2-2\,ab\,\cos C$, and solving it for $\cos C$, we get

$$\cos C=\frac{c^2-a^2-b^2}{-2\,ab}$$

Multiplying the numerator and denominator of the fraction by

-1 will get rid of the minus sign in the denominator without altering the value of the fraction. Then

$$\cos C = \frac{a^2+b^2-c^2}{2\,ab}$$

If the fraction comes out negative it indicates that the angle is greater than 90°.

In the same way we can derive formulas for cos A and cos B in terms of the sides, giving us the three formulas:

$$\cos A = \frac{b^2+c^2-a^2}{2\,bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2\,ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2\,ab}$$

Example:

Three studs are to be located on a plate in the relation shown in Fig. 209, so that their distances from each other are 3 in., $2\frac{1}{2}$ in., and $1\frac{1}{2}$ in. Find the height of the stud at C above the line c connecting A and B.

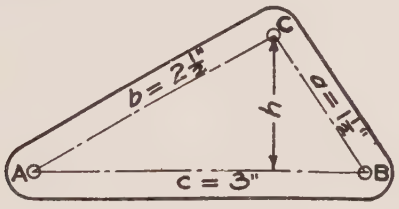


Fig. 209.

To get this height h we must first find the angle at A; we can then find h from the relation, $h = b \sin A$.

$$\cos A = \frac{b^2+c^2-a^2}{2\,bc}$$

$$= \frac{2\frac{1}{2}^2+3^2-1\frac{1}{2}^2}{2\times 2\frac{1}{2}\times 3} = \frac{6.25+9-2.25}{15} = \frac{13}{15} = .8666$$

$$A = \cos^{-1} .8666 = 29°\ 56'$$

$$h = b \sin A$$

$$= 2.5 \times \sin 29°\ 56' = 2.5 \times .4990 = 1.2485 \text{ in., } \textit{Answer.}$$

138. Law of Sines.—There may be cases of triangles where we have just one side to start with and have two or three angles. To get the other sides in such a case, we can use our law of sines, which is derived as follows: Let Fig. 210 represent any triangle with angles A, B, and C and the corresponding sides a, b, and c.

Draw a perpendicular from C to c and call this line h. Now we have two right triangles, one on either side of h. In one triangle $h = b \sin A$, while in the other $h = a \sin B$.
Therefore, $b \sin A = a \sin B$. (12)

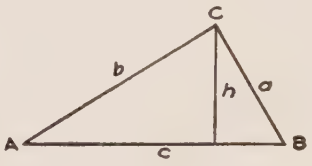


Fig. 210.

Dividing both sides of the equation by $\sin A$, we get:

$$b = \frac{a}{\sin A} \sin B$$

or, if we divide equation (12) by b, we get

$$\sin A = \frac{a \sin B}{b}$$

From these formulas we see that if we know two angles and the side opposite one of them we can get the side opposite the other angle; or if we know two sides and the angle opposite one side we can get the angle opposite the other side. This law is generally written in the following form:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Written in words this would be as follows: "*In any triangle, the sides are proportional to the sines of the angles opposite them.*"

139. Application of Laws to Problems.—There may be four possible cases of triangles to be solved, depending on what data is given.

Case I.—We may have two sides and the angle between them given.

Proceed as follows: Find the other side by using the law of cosines. Then, having all sides and one angle, find the other angles by using the law of sines.

Case II.—We may have two angles and the side between the two angles.

Proceed as follows: Get the third angle by subtracting the sum of the two given angles from 180°. Then use the law of sines to get the other two sides.

Case III.—We may have just the three sides given.

Proceed as follows: Using the law of cosines (transposed to get the angles) calculate each of the three angles.

Case IV.—We may have two sides given and the angle opposite one of the sides. From this information we might get two possible triangles. Suppose we have given b, a, and A in Fig. 211. As far as we know, the side a might lie at a or at a', because neither C nor c are known, and we might get either of the triangles shown, ABC or $AB'C$.

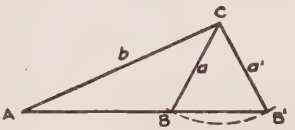


FIG. 211.

Proceed as follows: Having given a, b, and A, get sin B from the law of sines. By referring to a table of sines we naturally get the angle B' which is less than 90.° But this might as well be the sin B as sin B', for this reason: The angle B or $ABC = 180° - B'BC$ and, since $B'BC = B'$, we can see that $B = 180° - B'$. Now we saw in Art. 132 that the sine of any angle is the same as the sine of its supplement. Hence the value obtained for sin B would fit either triangle, ABC or $AB'C$.

Taking the two possible values of B we get two values for C, and from the law of sines we get two corresponding values of the side c.

140. Areas of Triangles.—If any of the angles of a triangle is known, the easiest way to get the area is to find an altitude from an angle and its adjacent side. Thus in Fig. 210, $h = b \sin A$ or $h = A \sin b$. From this, the area $= \frac{1}{2} cb \sin A$, or $= \frac{1}{2} ca \sin B$.

When only the three sides are known, we can derive a formula which will give the area in terms of the sides. We have used this before (Art. 74, Chapter XII) but the proof of the formula may be of interest at this time.

Referring to Fig. 210,

$$\text{Area} = \tfrac{1}{2} c \times h = \tfrac{1}{2} cb \sin A$$

The angle A is not known, but from Art. 131, Chapter XVII,

$$\sin A = \sqrt{1 - (\cos A)^2}$$

and we know from the law of cosines that

$$\cos A = \frac{c^2 + b^2 - a^2}{2\,bc}$$

Then,

$$\sin A = \sqrt{1 - \left(\frac{c^2 + b^2 - a^2}{2\,bc}\right)^2}$$

$$\text{Area} = \tfrac{1}{2} bc \sqrt{\frac{(2\,bc)^2 - (c^2 + b^2 - a^2)^2}{(2\,bc)^2}}$$

$$= \tfrac{1}{2} \not{b}\not{c} \frac{\sqrt{(2\,bc)^2-(c^2+b^2-a^2)^2}}{2\,\not{b}\not{c}}$$

$$= \tfrac{1}{4}\sqrt{(2\,bc)^2-(c^2+b^2-a^2)^2}$$

Under the radical sign we have the difference of the square of two quantities and we know, for instance, that $x^2-y^2=(x-y)(x+y)$. In like manner the above equation can be written

$$\text{Area} = \tfrac{1}{4}\sqrt{(2\,bc-c^2-b^2+a^2)(2\,bc+c^2+b^2-a^2)}$$

$$= \tfrac{1}{4}\sqrt{[a^2-(b^2-2\,bc+c^2)][(b^2+2\,bc+c^2)-a^2]}$$

$$= \tfrac{1}{4}\sqrt{[a^2-(b-c)^2][(b+c)^2-a^2]}$$

Each of the terms in the brackets can be rewritten on the same basis as above, giving

$$\text{Area} = \tfrac{1}{4}\sqrt{(a-b+c)(a+b-c)\,(b+c-a)\,(b+c+a)}$$

Now let $s = \dfrac{a+b+c}{2}$, or $2s=(a+b+c)$.

Then our equation can be written

$$\text{Area} = \tfrac{1}{4}\sqrt{(2\,s-2\,b)(2\,s-2\,c)(2\,s-2\,a)\,2\,s}$$

$$= \sqrt{\frac{2(s-b)\,2(s-c)\,2\,(s-a)2s}{16}}$$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the three sides of a triangle and s is one-half their sum.

PROBLEMS

183. Find, with the aid of the tables, the values of the following functions:

Sine, cosine, and tangent of 150°
Tangent and cotangent of 135°
Sine, cosine, and secant of 160° 35′.

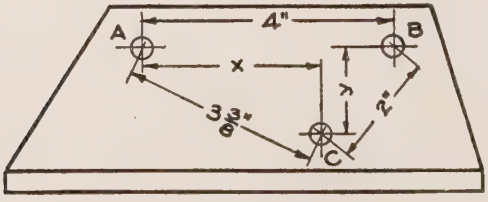


Fig. 212.

184. Three holes A, B, and C for gear studs are to be drilled in a plate as shown in Fig. 212. In order to locate them accurately, the work is to be done in a milling machine, the plate being clamped in a vertical position on the table. Hole B is drilled first; then the table is moved horizontally

4 in. and A is drilled. To locate C after drilling A, the table must be returned a distance x and then elevated a distance y. Calculate these distances x and y.

185. The sketch of Fig. 213 shows the floor plan of a building. It is proposed to rent out the corner ABC for a tin shop and, to order lumber

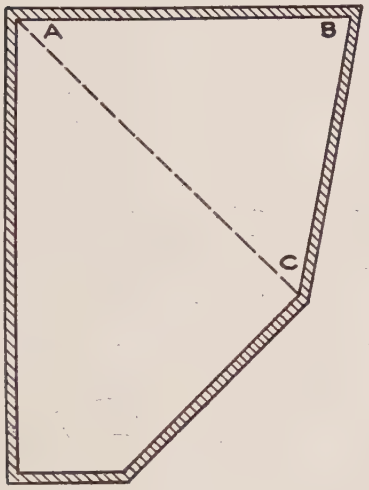


Fig. 213.

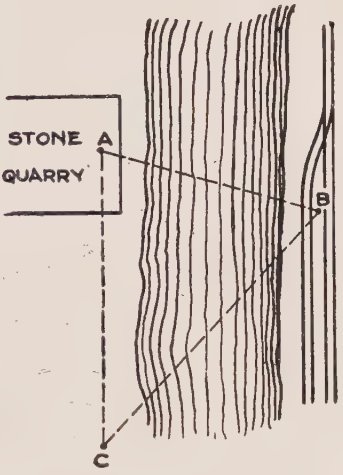


Fig. 214.

for the partition, it is necessary to know the length AC. It is not convenient at present to measure this distance because of certain obstructions, but we can get the sides AB and BC and the angle at B. If $AB = 160$ ft., $BC = 125$ ft., and angle $B = 64° 15'$, find the length of AC. Also calculate the area of floor space rented to the tin shop.

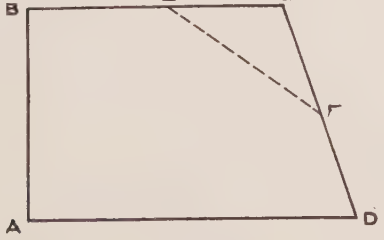


Fig. 215.

186. A stone quarry is located on the opposite side of a river from a railroad siding as shown in Fig. 214. To load the cars, it is proposed to build a cable tramway between two points A and B. The distance AC of 500 ft. is laid off and the angles at A and C found to be $74° 35'$ and $44° 15'$ respectively. Calculate the length of cable necessary for AB, allowing 30 it. for sag and fastenings.

187. Fig. 215 shows the boundaries of a proposed factory site. To measure the angle at C we lay off CE and CF each equal to 30 ft., and find that the distance EF is 48 ft. 6 in. What is the angle at C?

188. In the steam engine shown in Fig. 216 the stroke of the piston is 24 in.; the length of the connecting rod is 60 in.; and the length of the crank from center of shaft to center of crank pin is 12 in.

(a) Find the distance from the center of the crank shaft to the center of the wrist pin when, as shown in the sketch, the crank has turned 60° from the horizontal.

(b) Find how far the piston has travelled on its stroke when the crank is in this position?

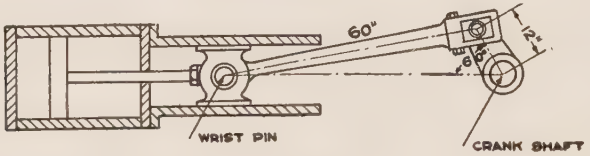


Fig. 216.

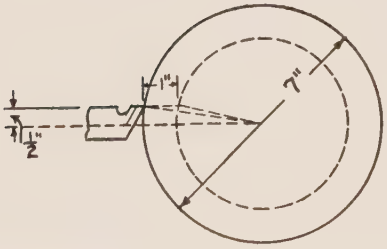


Fig. 217.

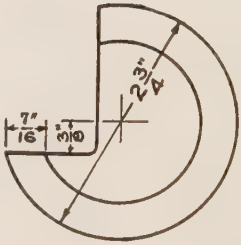


Fig. 218.

189. A machinist is turning a piece of 7-in. stock in a lathe and has his tool set $\frac{1}{2}$ in. above center, as shown in Fig. 217. What would be the diameter of the piece when the cross-slide carrying the tool has been fed in 1 in.?

190. A circular forming tool is to be made as shown in Fig. 218. What must be the diameter of the inner circle in order to leave the lip of $\frac{7}{16}$ in. on a line $\frac{3}{8}$ in. below the center of the circles?

CHAPTER XIX

LOGARITHMS

141. The Principle of Logarithms.—By the use of logarithms many problems involving multiplication, division, raising a number to a power, or extracting a root are worked much more rapidly than by ordinary arithmetical processes. Many calculations which are very difficult or almost impossible by the methods of ordinary arithmetic are readily performed by the help of logarithms.

The *Logarithm* of a number is the exponent of that power to which a certain fixed number must be raised to give the number first mentioned. The fixed number is called the *base* of the system; the most common system having the fixed number 10 as its base. Taking 10 as the base, the logarithm of 100 is 2, since 100 is 10^2; the logarithm of 1000 is 3, since 1000 is 10^3; the logarithm of 10,000 is 4, since 10,000 is 10^4; and so on.

In studying the multiplication of algebraic terms in Chapter V, we learned that we could multiply terms having the same base by adding the exponents. For example

$$x^2 \times x^5 = x^{2+5} = x^7$$

In the same manner

$$10^3 \times 10^6 = 10^{3+6} = 10^9$$

Probably many of us have unconsciously used this method in multiplying such numbers. In multiplying $1000 \times 1{,}000{,}000$ we know that there will be as many ciphers in the product as there are in both the numbers. We find 3 ciphers in 1000 and 6 in 1,000,000, so we write the product with 3+6 or 9 ciphers, giving as a result 1,000,000,000. All such numbers which are powers of the base 10 can be multiplied by adding the numbers of ciphers. In doing this we are really adding the logarithms of the numbers:

$$10 \times 1000 = 10^1 \times 10^3 = 10^4 = 10{,}000$$
$$100 \times 100 = 10^2 \times 10^2 = 10^4 = 10{,}000$$
$$10{,}000 \times 1000 = 10^4 \times 10^3 = 10^7 = 10{,}000{,}000$$

In like manner, division can be performed by substracting exponents:

$$10{,}000 \div 1000 = 10^4 \div 10^3 = 10^1 = 10$$
$$100{,}000 \div 100 = 10^5 \div 10^2 = 10^3 = 1000$$

142. Fractional Exponents.—To be able to apply the principles of logarithms to calculations involving other numbers than the exact powers of 10, we must understand how such numbers can be represented as powers of the base 10. If a number lies between 100 and 1000, it can be written as 10 with a decimal exponent somewhere between 2 and 3, since 100 is 10^2 and 1000 is 10^3. For example: 300 can be written as $10^{2.4771}$; 421 can be written as $10^{2.6243}$; 7290 can be written $10^{3.8627}$; etc. The decimal parts of these logarithms or exponents have been calculated and arranged in tables for our use. An idea of the meaning of these decimal exponents can be obtained from the following cases.

If we extract the square root of 1000, we get 31.62+. On the other hand, if we consider 1000 as 10^3, we will get $10^{1.5}$ as the square root, since $x^{1.5} \times x^{1.5} = x^3$. Then the logarithm of 31.62 is 1.5.

If we extract the square root of 10, we get 3.162. The square root of 10^1 is also $10^{.5}$. Hence the logarithm of 3.162 is .5.

The fourth root (square root of the square root) of 1000 is 5.624, while the fourth root of 10^3 is $10^{.75}$. Hence the logarithm of 5.624 is .75. Likewise, the logarithm of 56.24, which is the fourth root of 10,000,000 or 10^7, is 1.75.

143. Common Logarithms.—A system of logarithms might be worked out using any number as the base; but for many practical reasons the system using 10 as the base is the one in general use. This system is called the *common system* of logarithms (or the *Briggs* system, since it was developed by Henry Briggs). Logarithms having the base 10 are called *common logarithms.*

One great advantage of the common system of logarithms is that the whole number or integral part of the logarithm of a number can be seen by a brief inspection of the number. If a number lies between 100 and 1000, its logarithm will be 2 + a decimal; in other words, numbers having three figures to the left of the decimal point have a logarithm of 2.+. Likewise numbers having four figures in the integral (whole number) part have a logarithm of 3.+, and so on, the integral part of the

logarithm always being 1 less than the number of digits to the left of the decimal point in the number.

The whole number or integral part of a logarithm is called the *characteristic* of the logarithm.

The decimal part of the logarithm is called the *mantissa.*

If a number be resolved into two factors, of which one is an exact or integral power of 10, and the other lies between 1 and 10, then the exponent of 10 is the characteristic and the logarithm of the other factor is the mantissa. For example, 300 can be resolved into 100×3, or $10^2 \times 3$. The logarithm of 300 is therefore 2.+ log 3. The logarithm of 3 is .4771; hence, the logarithm of 300 is 2.4771. Likewise, the logarithm of 3000 is 3.4771, and the logarithm of 30,000 is 4.4771.

From this we see that the logarithms of numbers that differ only in the position of the decimal point have the same mantissa, but have different characteristics. For example: log $4 = .6021$; log $40 = 1.6021$; log $400 = 2.6021$; log $4000 = 3.6021$. Consequently, in a table of common logarithms we need to give only the mantissas, since the characteristics can be observed by the position of the decimal point in the number.

144. Explanation of Logarithm Tables.—On pages 195 to 200 will be found a four-place table of common logarithms. In the columns headed *no.* are given the numbers; in the column to the right of this, under *log* are given the mantissas. The figures in the *no.* column are for corresponding numbers regardless of where the decimal point may be in the number. For example, opposite the number 124 we find the mantissa .0934. This is the mantissa for the logarithm of .124 or 1.24 or 12.4 or 12,400, etc. The characteristic is obtained by inspection of the number, being 1 less than the number of figures to the left of the decimal point. Thus, log $1.24 = .0934$; log $12.4 = 1.0934$; log $124 = 2.0934$; log $12{,}400 = 4.0934$.

The student may check the following logarithms from the tables to see if the method is clearly understood: Log $112 = 2.0492$; log $288 = 2.4594$; log $4.95 = 0.6946$; log $11.37 = 1.0558$; log $97{,}600 = 4.9894$.

The logarithm of the base 10 is 1 because 10 is 10^1.

The logarithm of 1 is 0. To show this let us divide 10 by 10. The quotient is, of course, 1. If we divide 10^1 by 10^1 by the algebraic method of subtracting exponents, we get $10^1 \div 10^1 = 10^{1-1} = 10^0 = 1$. (In the same way any other number to the zero

power can be shown to be 1.) Hence, log 1 = 0. This will also be seen if we remember that multiplying any number by 1 gives the same number. If we multiply any number by 1, by the system of adding logarithms we should get the same logarithm for the product as for the number. This also shows that the log 1 must be zero.

145. Interpolation from the Tables.—The tables only give numbers of three figures. To find the logarithms of longer numbers we use the process of interpolation. For example, we will find the logarithm of 23.64. We first look in the table for the number 236 and find that the mantissa of log 236 = .3729. The mantissa for 2364 will lie between that for 236 and that for 237. Just at the right of the *log* columns in the tables is a column headed *d*. This column gives the differences between the mantissas just at the left. The difference between the mantissas of the numbers 236 and 237 is given as 18. This can also be found by subtracting 3729 from 3747. For 2364 we must add .4 of this difference, or .4 of 18 = 7.+ to the mantissa for 236. Hence we have

$$\begin{aligned} \text{mantissa of log } 236 &= .3729 \\ \text{correction for } 2364 &= \quad 7 \\ \hline \text{mantissa of log of } 2364 &= .3736 \\ \text{characteristic of log of } 23.64 &= 1. \\ \hline \log 23.64 &= 1.3736 \end{aligned}$$

The work of finding logarithms of numbers is really very short. If a number has three figures, we can write down its characteristic at once from the location of the decimal point and can affix the mantissa taken directly from the table. The following examples can be checked from the tables for practice.

$$\begin{aligned} \log \quad 3.76 &= 0.5752 \\ \log \quad 9760. &= 3.9894 \\ \log \quad 52.30 &= 1.7185 \\ \log \ 76{,}000 &= 4.8808 \\ \log \quad 291. &= 2.4639 \\ \log \ 27{,}825 &= 4.4444 \end{aligned}$$

To facilitate finding the logarithms of numbers of four figures, we have the column of differences, while at the right of each page under the heading *pp* are the *proportional parts* of these differences. This enables us to get a correction at once. The decimal part of the correction may be carried if desired, but it

is of no value since the table only gives mantissas to four figures. Hence, it is better to drop the decimal part of the correction or raise it to the next higher whole number if the decimal part is .5 or more.

To illustrate we will find the logarithm of 1567.

mantissa for 156 =	.1931
difference given as 28; correction for 7 in the column headed 28 is 19.6, or	20
mantissa for 1567 =	.1951
characteristic for 1567 =	3.
logarithm of 1567 =	3.1951

The addition of the correction can usually be done mentally while writing down the logarithm, so that the corrected logarithm can be written at once. The following examples may be checked from the tables for practice.

log 1526 = 3.1835
log 29.35 = 1.4676
log 6.571 = 0.8177
log 426.8 = 2.6302

It is useless to try to use the tables for numbers of more than four figures, so that any figures after the fourth may as well be dropped. In getting log of 24.628, we would get the log of 24.63; the log of 496.22 would be obtained as the log of 496.2; etc. A four-place log table can be used for calculations dealing with numbers of four figures and will give the result accurately to the fourth figure. This is accurate enough for nearly all practical purposes. When greater accuracy is desired, tables giving logarithms to 5, 6, 7, and even more places can be found in handbooks. These will give results accurately to as many figures as there are decimal places in the logarithms given.

146. Antilogarithms.—After we have worked a problem by logarithms we have the log of the answer and the last step is to find the number of which this is the logarithm. This is called *finding the antilogarithm.* To do this, first take the mantissa of the answer and find in the table the number corresponding to it. If the exact mantissa does not appear in the table, find the nearest value *less* than the given mantissa. Subtract this

value found in the table from the given mantissa, and find this difference in the proper column of corrections at the right of the page. The corresponding figure found in the column of *pp* should be joined to the number found from the table, giving the result to four figures.

Then point off the answer by means of the characteristic. Remember that the characteristic is 1 *less* than the number of figures in the integral part of the number.

Example:

Find the antilog of 2.6639, or in other words, find the number of which 2.6639 is the logarithm.

Taking the mantissa .6639, we find the nearest mantissa given in the table less than .6639 to be .6637; this is the mantissa of 461. Subtracting .6637 from .6639 we get a difference of 2. In the column headed *d* we find 9 as the difference between .6637 and .6646, the two values in the table between which our mantissa lies. In the column headed *pp* we look in the table for a difference of 9. Under 9 the nearest value to 2 is 1.8 which corresponds to a fourth figure of 2 (in the left-hand column). Therefore, .6639 is the mantissa of 4612.

We now point off our answer by use of the characteristic, which was 2. This shows that there will be 3 figures in the integral part of our answer and, therefore, we have 461.2.

Hence, antilog 2.6639 = 461.2.

147. Multiplication by Logarithms.—Suppose x and y represent two numbers that are to be multiplied together. If a represents the log of x and b represents the log of y, then

$$x = 10^a \text{ and } y = 10^b$$
$$xy = 10^a \times 10^b = 10^{a+b}$$
$$\text{or, } \log xy = a + b$$
$$\text{or, } \log xy = \log x + \log y.$$

In words, this shows that the logarithm of the product of two numbers is the sum of the logarithms of the numbers.

Example 1:

Multiply 47.61 by 37.65.

log 47.61 = 1.6777
log 37.65 = 1.5758
log of product = 3.2535
antilog 3.2535 = 1792+
Hence $47.61 \times 37.56 = 1792+$, *Answer.*

Explanation: The mantissa for 476 is .6776. The correction for a fourth figure of 1 is .9 or 1., giving .6777, and the characteristic is 1. The mantissa for 376 is .5752, and the correction for a fourth figure of 5 is 5.5, or 6, giving .5758, and the characteristic is 1. Adding the logs we get 3.2535 as the log of the product. The mantissa next less than 2535 is 2529 which corresponds to the number 179. Our mantissa is 6 more than 2529. Opposite 2529 we find the difference is 24. In the column of *pp* headed 24 we find 4.8, corresponding to a fourth figure of 2, and 7.2 corresponding to a fourth figure of 3. Our result is between these, so the product is 1792.+ and probably is about 1792.5 (by actual multiplication this proves to be 1792.5165).

Example 2:

Find the product of $1.75 \times 3.142 \times 1.625$.

log 1.75	$=0.2430$
log 3.142	$=0.4972$
log 1.625	$=0.2109$
log of product	$=0.9511$
antilog 0.9511	$=8.935+$

Hence, $1.75 \times 3.142 \times 1.625 = 8.935+$, *Answer.*

Example 3:

Find the product of 3.142×2.875^2.

log 3.142	$=0.4972$
log 2.875	$=0.4587$
log 2.875	$=0.4587$
log of product	$=1.4146$
antilog of 1.4146	$=25.98+$

Hence, $3.142 \times 2.875^2 = 25.98+$, *Answer.*

148. Division by Logarithms.—According to the principle of logarithms we can divide one number by another by subtracting the log of the divisor from the log of the dividend.

$$\text{If } x = 10^a \text{ and } y = 10^b$$
$$x \div y = 10^a \div 10^b = 10^{a-b}$$
$$\text{then } \log (x \div y) = a - b = \log x - \log y$$

In dividing by logarithms we look up the log of the dividend, then that of the divisor, and subtract the latter from the former. We then look up the antilog of the difference in order to get the quotient.

Example 1:

Divide 113.1 by 3.142

log 113.1	$=2.0535$
log 3.142	$=0.4972$
log of quotient	$=1.5563$
antilog 1.5563	$=36.00$

Hence, $113.1 \div 3.142 = 36$, *Answer.*

Example 2:

Find the value of $\dfrac{1.725 \times 296 \times 37.5}{429 \times 24}$

Here we can add the logs of the terms in the numerator to find the log of the product; in like manner we find the log of the denominator; we then subtract to find the log of the quotient.

log 1.725	$=0.2368$	log 429	$=2.6325$
log 296	$=2.4713$	log 24	$=1.3802$
log 37.5	$=1.5740$	log of denominator	$=4.0127$
log of numerator	$=4.2821$		
log of denominator	$=4.0127$		
log of quotient	$=0.2694$		
antilog 0.2694	$=1.86$		

Hence, $\dfrac{1.725 \times 296 \times 37.5}{429 \times 24} = 1.86$, *Answer.*

PROBLEMS

Note.—Use logarithms wherever possible.

191. Find the logarithms of the following numbers:

(a) 12.6; (b) 1.38; (c) 4,826; (d) 279,000; (e) 49,237; (f) 123.4; (g) 2,795,000,000.

192. Find the antilogarithms of the following logarithms:

(a) 2.7240; (b) 0.4239; (c) 6.2780; (d) 4.0172; (e) 0.7364; (f) 5.2173; (g) 8.2760.

193. Multiply 7582 by 3791. *Answer,* 28,740,000

194. Multiply $382.1 \times 74.83 \times 2743$.

195. Divide 345.8 by 86.27.

196. Find the value of $\dfrac{4763 \times 2896 \times 2872}{8641 \times 738.6 \times 564}$

197. In taking inventory, we find 2320 castings from one pattern, each weighing 6.38 lb. If they are worth 1.875 cents a pound, what is the total value of the castings?

198. What is the value of 839 linear feet of steel bars weighing 6.38 lb. per foot, if steel is worth $31.25 per ton of 2000 lb.

199. The horse-power of a 12-in. × 16-in. gas engine at 260 R. P. M. with a mean effective pressure of 81.5 lb. per square inch is represented by the equation:

$$\text{H. P.} = \frac{81.5 \times 1.333 \times 113.1 \times 130}{33{,}000}$$

Calculate the horse-power of this engine.

200. The horse-power of a boiler is determined by the number of square feet of metal surface having hot gases on one side and water on the other. Figuring 10 sq. ft. per horse-power, the horse-power of a water-tube boiler having 112 tubes $3\frac{1}{2}$ in. in diameter and 188 in. long would be:

$$\text{H. P.} = \frac{112 \times 3.142 \times 3.5 \times 188}{10 \times 144}$$

Calculate the horse-power of this boiler.

LOGARITHMS OF NUMBERS.

no.	log.	d.	no.	log.	d.	no.	log.	d.	no.	log.	d.
100	0000	43	150	1761	29	200	3010	22	250	3979	18
101	0043	43	151	1790	28	201	3032	22	251	3997	17
102	0086	42	152	1818	29	202	3054	21	252	4014	17
103	0128	42	153	1847	28	203	3075	21	253	4031	17
104	0170	42	154	1875	28	204	3096	22	254	4048	17
105	0212	41	155	1903	28	205	3118	21	255	4065	17
106	0253	41	156	1931	28	206	3139	21	256	4082	17
107	0294	40	157	1959	28	207	3160	21	257	4099	17
108	0334	40	158	1987	27	208	3181	20	258	4116	17
109	0374	40	159	2014	27	209	3201	21	259	4133	17
110	0414	39	160	2041	27	210	3222	21	260	4150	16
111	0453	39	161	2068	27	211	3243	20	261	4166	17
112	0492	39	162	2095	27	212	3263	21	262	4183	17
113	0531	38	163	2122	26	213	3284	20	263	4200	16
114	0569	38	164	2148	27	214	3304	20	264	4216	16
115	0607	38	165	2175	26	215	3324	21	265	4232	17
116	0645	37	166	2201	26	216	3345	20	266	4249	16
117	0682	37	167	2227	26	217	3365	20	267	4265	16
118	0719	36	168	2253	26	218	3385	19	268	4281	17
119	0755	37	169	2279	25	219	3404	20	269	4298	16
120	0792	36	170	2304	26	220	3424	20	270	4314	16
121	0828	36	171	2330	25	221	3444	20	271	4330	16
122	0864	35	172	2355	25	222	3464	19	272	4346	16
123	0899	35	173	2380	25	223	3483	19	273	4362	16
124	0934	35	174	2405	25	224	3502	20	274	4378	15
125	0969	35	175	2430	25	225	3522	19	275	4393	16
126	1004	34	176	2455	25	226	3541	19	276	4409	16
127	1038	34	177	2480	24	227	3560	19	277	4425	15
128	1072	34	178	2504	25	228	3579	19	278	4440	16
129	1106	33	179	2529	24	229	3598	19	279	4456	16
130	1139	34	180	2553	24	230	3617	19	280	4472	15
131	1173	33	181	2577	24	231	3636	19	281	4487	15
132	1206	33	182	2601	24	232	3655	19	282	4502	16
133	1239	32	183	2625	23	233	3674	18	283	4518	15
134	1271	32	184	2648	24	234	3692	19	284	4533	15
135	1303	32	185	2672	23	235	3711	18	285	4548	16
136	1335	32	186	2695	23	236	3729	18	286	4564	15
137	1367	32	187	2718	24	237	3747	19	287	4579	15
138	1399	31	188	2742	23	238	3766	18	288	4594	15
139	1430	31	189	2765	23	239	3784	18	289	4609	15
140	1461	31	190	2788	22	240	3802	18	290	4624	15
141	1492	31	191	2810	23	241	3820	18	291	4639	15
142	1523	30	192	2833	23	242	3838	18	292	4654	15
143	1553	31	193	2856	22	243	3856	18	293	4669	14
144	1584	30	194	2878	22	244	3874	18	294	4683	15
145	1614	30	195	2900	23	245	3892	17	295	4698	15
146	1644	29	196	2923	22	246	3909	18	296	4713	15
147	1673	30	197	2945	22	247	3927	18	297	4728	14
148	1703	29	198	2967	22	248	3945	17	298	4742	15
149	1732	29	199	2989	21	249	3962	17	299	4757	14
150	1761		200	3010		250	3979		300	4771	

pp.

	43	42	41
1	4.3	4.2	4.1
2	8.6	8.4	8.2
3	12.9	12.6	12.3
4	17.2	16.8	16.4
5	21.5	21.0	20.5
6	25.8	25.2	24.6
7	30.1	29.4	28.7
8	34.4	33.6	32.8
9	38.7	37.8	36.9

	40	39	38
1	4.0	3.9	3.8
2	8.0	7.8	7.6
3	12.0	11.7	11.4
4	16.0	15.6	15.2
5	20.0	19.5	19.0
6	24.0	23.4	22.8
7	28.0	27.3	26.6
8	32.0	31.2	30.4
9	36.0	35.1	34.2

	37	36	35
1	3.7	3.6	3.5
2	7.4	7.2	7.0
3	11.1	10.8	10.5
4	14.8	14.4	14.0
5	18.5	18.0	17.5
6	22.2	21.6	21.0
7	25.9	25.2	24.5
8	29.6	28.8	28.0
9	33.3	32.4	31.5

	34	33	32
1	3.4	3.3	3.2
2	6.8	6.6	6.4
3	10.2	9.9	9.6
4	13.6	13.2	12.8
5	17.0	16.5	16.0
6	20.4	19.8	19.2
7	23.8	23.1	22.4
8	27.2	26.4	25.6
9	30.6	29.7	28.8

	31	30	29
1	3.1	3.0	2.9
2	6.2	6.0	5.8
3	9.3	9.0	8.7
4	12.4	12.0	11.6
5	15.5	15.0	14.5
6	18.6	18.0	17.4
7	21.7	21.0	20.3
8	24.8	24.0	23.2
9	27.9	27.0	26.1

	28	27	26
1	2.8	2.7	2.6
2	5.6	5.4	5.2
3	8.4	8.1	7.8
4	11.2	10.8	10.4
5	14.0	13.5	13.0
6	16.8	16.2	15.6
7	19.6	18.9	18.2
8	22.4	21.6	20.8
9	25.2	24.3	23.4

LOGARITHMS OF NUMBERS.

no.	log.	d.	no.	log.	d.	no.	log.	d.	no.	log.	d.
300	4771		350	5441		400	6021		450	6532	
		15			12			10			10
301	4786		351	5453		401	6031		451	6542	
		14			12			11			9
302	4800		352	5465		402	6042		452	6551	
		14			13			11			10
303	4814		353	5478		403	6053		453	6561	
		15			12			11			10
304	4829		354	5490		404	6064		454	6571	
		14			12			11			9
305	4843		355	5502		405	6075		455	6580	
		14			12			10			10
306	4857		356	5514		406	6085		456	6590	
		14			13			11			9
307	4871		357	5527		407	6096		457	6599	
		15			12			11			10
308	4886		358	5539		408	6107		458	6609	
		14			12			10			9
309	4900		359	5551		409	6117		459	6618	
		14			12			11			10
310	4914		360	5563		410	6128		460	6628	
		14			12			10			9
311	4928		361	5575		411	6138		461	6637	
		14			12			11			9
312	4942		362	5587		412	6149		462	6646	
		13			12			11			10
313	4955		363	5599		413	6160		463	6656	
		14			12			10			9
314	4969		364	5611		414	6170		464	6665	
		14			12			10			10
315	4983		365	5623		415	6180		465	6675	
		14			12			11			9
316	4997		366	5635		416	6191		466	6684	
		14			12			10			9
317	5011		367	5647		417	6201		467	6693	
		13			11			11			9
318	5024		368	5658		418	6212		468	6702	
		14			12			10			10
319	5038		369	5670		419	6222		469	6712	
		13			12			10			9
320	5051		370	5682		420	6232		470	6721	
		14			12			11			9
321	5065		371	5694		421	6243		471	6730	
		14			11			10			9
322	5079		372	5705		422	6253		472	6739	
		13			12			10			10
323	5092		373	5717		423	6263		473	6749	
		13			12			11			9
324	5105		374	5729		424	6274		474	6758	
		14			11			10			9
325	5119		375	5740		425	6284		475	6767	
		13			12			10			9
326	5132		376	5752		426	6294		476	6776	
		13			11			10			9
327	5145		377	5763		427	6304		477	6785	
		14			12			10			9
328	5159		378	5775		428	6314		478	6794	
		13			11			11			9
329	5172		379	5786		429	6325		479	6803	
		13			12			10			9
330	5185		380	5798		430	6335		480	6812	
		13			11			10			9
331	5198		381	5809		431	6345		481	6821	
		13			12			10			9
332	5211		382	5821		432	6355		482	6830	
		13			11			10			9
333	5224		383	5832		433	6365		483	6839	
		13			11			10			9
334	5237		384	5843		434	6375		484	6848	
		13			12			10			9
335	5250		385	5855		435	6385		485	6857	
		13			11			10			9
336	5263		386	5866		436	6395		486	6866	
		13			11			10			9
337	5276		387	5877		437	6405		487	6875	
		13			11			10			9
338	5289		388	5888		438	6415		488	6884	
		13			11			10			9
339	5302		389	5899		439	6425		489	6893	
		13			12			10			9
340	5315		390	5911		440	6435		490	6902	
		13			11			9			9
341	5328		391	5922		441	6444		491	6911	
		12			11			10			9
342	5340		392	5933		442	6454		492	6920	
		13			11			10			8
343	5353		393	5944		443	6464		493	6928	
		13			11			10			9
344	5366		394	5955		444	6474		494	6937	
		12			11			10			9
345	5378		395	5966		445	6484		495	6946	
		13			11			9			9
346	5391		396	5977		446	6493		496	6955	
		12			11			10			9
347	5403		397	5988		447	6503		497	6964	
		13			11			10			8
348	5416		398	5999		448	6513		498	6972	
		12			11			9			9
349	5428		399	6010		449	6522		499	6981	
		13			11			10			9
350	5441		400	6021		450	6532		500	6990	

pp.

	25	24	23
1	2.5	2.4	2.3
2	5.0	4.8	4.6
3	7.5	7.2	6.9
4	10.0	9.6	9.2
5	12.5	12.0	11.5
6	15.0	14.4	13.8
7	17.5	16.8	16.1
8	20.0	19.2	18.4
9	22.5	21.6	20.7

	22	21	20
1	2.2	2.1	2.0
2	4.4	4.2	4.0
3	6.6	6.3	6.0
4	8.8	8.4	8.0
5	11.0	10.5	10.0
6	13.2	12.6	12.0
7	15.4	14.7	14.0
8	17.6	16.8	16.0
9	19.8	18.9	18.0

	19	18	17
1	1.9	1.8	1.7
2	3.8	3.6	3.4
3	5.7	5.4	5.1
4	7.6	7.2	6.8
5	9.5	9.0	8.5
6	11.4	10.8	10.2
7	13.3	12.6	11.9
8	15.2	14.4	13.6
9	17.1	16.2	15.3

	16	15	14
1	1.6	1.5	1.4
2	3.2	3.0	2.8
3	4.8	4.5	4.2
4	6.4	6.0	5.6
5	8.0	7.5	7.0
6	9.6	9.0	8.4
7	11.2	10.5	9.8
8	12.8	12.0	11.2
9	14.4	13.5	12.6

	13	12	11
1	1.3	1.2	1.1
2	2.6	2.4	2.2
3	3.9	3.6	3.3
4	5.2	4.8	4.4
5	6.5	6.0	5.5
6	7.8	7.2	6.6
7	9.1	8.4	7.7
8	10.4	9.6	8.8
9	11.7	10.8	9.9

	10	9	8
1	1.0	0.9	0.8
2	2.0	1.8	1.6
3	3.0	2.7	2.4
4	4.0	3.6	3.2
5	5.0	4.5	4.0
6	6.0	5.4	4.8
7	7.0	6.3	5.6
8	8.0	7.2	6.4
9	9.0	8.1	7.2

LOGARITHMS OF NUMBERS.

no.	log.	d.	no.	log.	d.	no.	log.	d.	no.	log.	d.
500	6990	8	550	7404	8	600	7782	7	650	8129	7
501	6998	9	551	7412	7	601	7789	7	651	8136	6
502	7007	9	552	7419	8	602	7796	7	652	8142	7
503	7016	8	553	7427	8	603	7803	7	653	8149	7
504	7024	9	554	7435	8	604	7810	8	654	8156	6
505	7033	9	555	7443	8	605	7818	7	655	8162	7
506	7042	8	556	7451	8	606	7825	7	656	8169	7
507	7050	9	557	7459	7	607	7832	7	657	8176	6
508	7059	8	558	7466	8	608	7839	7	658	8182	7
509	7067	9	559	7474	8	609	7846	7	659	8189	6
510	7076	8	560	7482	8	610	7853	7	660	8195	7
511	7084	9	561	7490	7	611	7860	8	661	8202	7
512	7093	8	562	7497	8	612	7868	7	662	8209	6
513	7101	9	563	7505	8	613	7875	7	663	8215	7
514	7110	8	564	7513	7	614	7882	7	664	8222	6
515	7118	8	565	7520	8	615	7889	7	665	8228	7
516	7126	9	566	7528	8	616	7896	7	666	8235	6
517	7135	8	567	7536	7	617	7903	7	667	8241	7
518	7143	9	568	7543	8	618	7910	7	668	8248	6
519	7152	8	569	7551	8	619	7917	7	669	8254	7
520	7160	8	570	7559	7	620	7924	7	670	8261	6
521	7168	9	571	7566	8	621	7931	7	671	8267	7
522	7177	8	572	7574	8	622	7938	7	672	8274	6
523	7185	8	573	7582	7	623	7945	7	673	8280	7
524	7193	9	574	7589	8	624	7952	7	674	8287	6
525	7202	8	575	7597	7	625	7959	7	675	8293	6
526	7210	8	576	7604	8	626	7966	7	676	8299	7
527	7218	8	577	7612	7	627	7973	7	677	8306	6
528	7226	9	578	7619	8	628	7980	7	678	8312	7
529	7235	8	579	7627	7	629	7987	6	679	8319	6
530	7243	8	580	7634	8	630	7993	7	680	8325	6
531	7251	8	581	7642	7	631	8000	7	681	8331	7
532	7259	8	582	7649	8	632	8007	7	682	8338	6
533	7267	8	583	7657	7	633	8014	7	683	8344	7
534	7275	9	584	7664	8	634	8021	7	684	8351	6
535	7284	8	585	7672	7	635	8028	7	685	8357	6
536	7292	8	586	7679	7	636	8035	6	686	8363	7
537	7300	8	587	7686	8	637	8041	7	687	8370	6
538	7308	8	588	7694	7	638	8048	7	688	8376	6
539	7316	8	589	7701	8	639	8055	7	689	8382	6
540	7324	8	590	7709	7	640	8062	7	690	8388	7
541	7332	8	591	7716	7	641	8069	6	691	8395	6
542	7340	8	592	7723	8	642	8075	7	692	8401	6
543	7348	8	593	7731	7	643	8082	7	693	8407	7
544	7356	8	594	7738	7	644	8089	7	694	8414	6
545	7364	8	595	7745	7	645	8096	6	695	8420	6
546	7372	8	596	7752	8	646	8102	7	696	8426	6
547	7380	8	597	7760	7	647	8109	7	697	8432	7
548	7388	8	598	7767	7	648	8116	6	698	8439	6
549	7396	8	599	7774	8	649	8122	7	699	8445	6
550	7404		600	7782		650	8129		700	8451	

pp.		
	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2
	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
7	5.6	4.8
9	6.3	5.4

LOGARITHMS OF NUMBERS.

no.	log.	d.	no.	log.	d.	no.	log.	d.	no.	log.	d.	pp.
700	8451	6	750	8751	5	800	9031	5	850	9294	5	
701	8457	6	751	8756	6	801	9036	6	851	9299	5	
702	8463	7	752	8762	6	802	9042	5	852	9304	5	
703	8470	6	753	8768	6	803	9047	6	853	9309	6	**7**
704	8476	6	754	8774	5	804	9053	5	854	9315	5	1 | 0.7
705	8482	6	755	8779	6	805	9058	5	855	9320	5	2 | 1.4
706	8488	6	756	8785	6	806	9063	6	856	9325	5	3 | 2.1
707	8494	6	757	8791	6	807	9069	5	857	9330	5	4 | 2.8
708	8500	6	758	8797	5	808	9074	5	858	9335	5	5 | 3.5
709	8506	7	759	8802	6	809	9079	6	859	9340	5	6 | 4.2
710	8513	6	760	8808	6	810	9085	5	860	9345	5	7 | 4.9
711	8519	6	761	8814	6	811	9090	6	861	9350	5	8 | 5.0
712	8525	6	762	8820	5	812	9096	5	862	9355	5	9 | 6.3
713	8531	6	763	8825	6	813	9101	5	863	9360	5	
714	8537	6	764	8831	6	814	9106	6	864	9365	5	
715	8543	6	765	8837	5	815	9112	5	865	9370	5	
716	8549	6	766	8842	6	816	9117	5	866	9375	5	**6**
717	8555	6	767	8848	6	817	9122	6	867	9380	5	1 | 0.6
718	8561	6	768	8854	5	818	9128	5	868	9385	5	2 | 1.2
719	8567	6	769	8859	6	819	9133	5	869	9390	5	3 | 1.8
720	8573	6	770	8865	6	820	9138	5	870	9395	5	4 | 2.4
721	8579	6	771	8871	5	821	9143	6	871	9400	5	5 | 3.0
722	8585	6	772	8876	6	822	9149	5	872	9405	5	6 | 3.6
723	8591	6	773	8882	5	823	9154	5	873	9410	5	7 | 4.2
724	8597	6	774	8887	6	824	9159	6	874	9415	5	8 | 4.8
725	8603	6	775	8893	5	825	9165	5	875	9420	5	9 | 5.4
726	8609	6	776	8899	6	826	9170	5	876	9425	5	
727	8615	6	777	8904	6	827	9175	5	877	9430	5	
728	8621	6	778	8910	5	828	9180	6	878	9435	5	
729	8627	6	779	8915	6	829	9186	5	879	9440	5	**5**
730	8633	6	780	8921	6	830	9191	5	880	9445	5	1 | 0.5
731	8639	6	781	8927	5	831	9196	5	881	9450	5	2 | 1.0
732	8645	6	782	8932	6	832	9201	5	882	9455	5	3 | 1.5
733	8651	6	783	8938	5	833	9206	6	883	9460	5	4 | 2.0
734	8657	6	784	8943	6	834	9212	5	884	9465	4	5 | 2.5
735	8663	6	785	8949	5	835	9217	5	885	9469	5	6 | 3.0
736	8669	6	786	8954	6	836	9222	5	886	9474	5	7 | 3.5
737	8675	6	787	8960	5	837	9227	5	887	9479	5	8 | 4.0
738	8681	5	788	8965	6	838	9232	6	888	9484	5	9 | 4.5
739	8686	6	789	8971	5	839	9238	5	889	9489	5	
740	8692	6	790	8976	6	840	9243	5	890	9494	5	
741	8698	6	791	8982	5	841	9248	5	891	9499	5	**4**
742	8704	6	792	8987	6	842	9253	5	892	9504	5	1 | 0.4
743	8710	6	793	8993	5	843	9258	5	893	9509	5	2 | 0.8
744	8716	6	794	8998	6	844	9263	5	894	9513	4	3 | 1.2
745	8722	6	795	9004	5	845	9269	6	895	9518	5	4 | 1.6
746	8727	5	796	9009	6	846	9274	5	896	9523	5	5 | 2.0
747	8733	6	797	9015	5	847	9279	5	897	9528	5	6 | 2.4
748	8739	6	798	9020	5	848	9284	5	898	9533	5	7 | 2.8
749	8745	6	799	9025	6	849	9289	5	899	9538	5	8 | 3.2
750	8751		800	9031		850	9294	5	900	9542	4	9 | 3.6

LOGARITHMS OF NUMBERS.

no.	log.	d.	no.	log.	d.	no.	log.	d.	no.	log.	d.	pp.
900	9542	5	950	9777	5	1000	0000	4	1050	0212	4	
901	9547	5	951	9782	4	1001	0004	5	1051	0216	4	
902	9552	5	952	9786	5	1002	0009	4	1052	0220	4	
903	9557	5	953	9791	4	1003	0013	4	1053	0224	4	
904	9562	4	954	9795	5	1004	0017	5	1054	0228	5	
905	9566	5	955	9800	5	1005	0022	4	1055	0233	4	
906	9571	5	956	9805	4	1006	0026	4	1056	0237	4	
907	9576	5	957	9809	5	1007	0030	5	1057	0241	4	
908	9581	5	958	9814	4	1008	0035	4	1058	0245	4	
909	9586	4	959	9818	5	1009	0039	4	1059	0249	4	
910	9590	5	960	9823	4	1010	0043	5	1060	0253	4	
911	9595	5	961	9827	5	1011	0048	4	1061	0257	4	**5**
912	9600	5	962	9832	4	1012	0052	4	1062	0261	4	1 0.5
913	9605	4	963	9836	5	1013	0056	4	1063	0265	4	2 1.0
914	9609	5	964	9841	4	1014	0060	5	1064	0269	4	3 1.5
915	9614	5	965	9845	5	1015	0065	4	1065	0273	5	4 2.0
916	9619	5	966	9850	4	1016	0069	4	1066	0278	4	5 2.5
917	9624	4	967	9854	5	1017	0073	4	1067	0282	4	6 3.0
918	9628	5	968	9859	4	1018	0077	4	1068	0286	4	7 3.5
919	9633	5	969	9863	5	1019	0082	5	1069	0290	4	8 4.0
920	9638	5	970	9868	4	1020	0086	4	1070	0294	4	9 4.5
921	9643	4	971	9872	5	1021	0090	4	1071	0298	4	
922	9647	5	972	9877	4	1022	0095	5	1072	0302	4	
923	9652	5	973	9881	5	1023	0099	4	1073	0306	4	
924	9657	4	974	9886	4	1024	0103	4	1074	0310	4	
925	9661	5	975	9890	4	1025	0107	4	1075	0314	4	
926	9666	5	976	9894	5	1026	0111	5	1076	0318	4	
927	9671	4	977	9899	4	1027	0116	4	1077	0322	4	
928	9675	5	978	9903	5	1028	0120	4	1078	0326	4	
929	9680	5	979	9908	4	1029	0124	4	1079	0330	4	
930	9685	4	980	9912	5	1030	0128	5	1080	0334	4	
931	9689	5	981	9917	4	1031	0133	4	1081	0338	4	
932	9694	5	982	9921	5	1032	0137	4	1082	0342	4	
933	9699	4	983	9926	4	1033	0141	4	1083	0346	4	**4**
934	9703	5	984	9930	4	1034	0145	4	1084	0350	4	1 0.4
935	9708	5	985	9934	5	1035	0149	5	1085	0354	4	2 0.8
936	9713	4	986	9939	4	1036	0154	4	1086	0358	4	3 1.2
937	9717	5	987	9943	5	1037	0158	4	1087	0362	4	4 1.6
938	9722	5	988	9948	4	1038	0162	4	1088	0366	4	5 2.0
939	9727	4	989	9952	4	1039	0166	4	1089	0370	4	6 2.4
940	9731	5	990	9956	5	1040	0170	5	1090	0374	4	7 2.8
941	9736	5	991	9961	4	1041	0175	4	1091	0378	4	8 3.2
942	9741	4	992	9965	4	1042	0179	4	1092	0382	4	9 3.6
943	9745	5	993	9969	5	1043	0183	4	1093	0386	4	
944	9750	4	994	9974	4	1044	0187	4	1094	0390	4	
945	9754	5	995	9978	5	1045	0191	4	1095	0394	4	
946	9759	4	996	9983	4	1046	0195	4	1096	0398	4	
947	9763	5	997	9987	4	1047	0199	5	1097	0402	4	
948	9768	5	998	9991	5	1048	0204	4	1098	0406	4	
949	9773	4	999	9996	4	1049	0208	4	1099	0410	4	
950	9777	4	1000	0000	4	1050	0212	4	1100	0414	4	

LOGARITHMS OF NUMBERS.

no.	log.	d.	no.	log.	d.	no.	log.	d.	no.	log.	d.
1100	0414		1150	0607		1200	0792		1250	0969	
1101	0418	4	1151	0611	4	1201	0795	3	1251	0973	4
1102	0422	4	1152	0615	4	1202	0799	4	1252	0976	3
1103	0426	4	1153	0618	3	1203	0803	4	1253	0980	4
		4			4			3			3
1104	0430		1154	0622		1204	0806		1254	0983	
1105	0434	4	1155	0626	4	1205	0810	4	1255	0986	3
1106	0438	4	1156	0630	4	1206	0813	3	1256	0990	4
		3			3			4			3
1107	0441		1157	0633		1207	0817		1257	0993	
1108	0445	4	1158	0637	4	1208	0821	4	1258	0997	4
1109	0449	4	1159	0641	4	1209	0824	3	1259	1000	3
		4			4			4			4
1110	0453		1160	0645		1210	0828		1260	1004	
1111	0457	4	1161	0648	3	1211	0831	3	1261	1007	3
1112	0461	4	1162	0652	4	1212	0835	4	1262	1011	4
1113	0465	4	1163	0656	4	1213	0839	4	1263	1014	3
		4			4			3			3
1114	0469		1164	0660		1214	0842		1264	1017	
1115	0473	4	1165	0663	3	1215	0846	4	1265	1021	4
1116	0477	4	1166	0667	4	1216	0849	3	1266	1024	3
		4			4			4			4
1117	0481		1167	0671		1217	0853		1267	1028	
1118	0484	3	1168	0674	3	1218	0856	3	1268	1031	3
1119	0488	4	1169	0678	4	1219	0860	4	1269	1035	4
		4			4			4			3
1120	0492		1170	0682		1220	0864		1270	1038	
1121	0496	4	1171	0686	4	1221	0867	3	1271	1041	3
1122	0500	4	1172	0689	3	1222	0871	4	1272	1045	4
1123	0504	4	1173	0693	4	1223	0874	3	1273	1048	3
		4			4			4			4
1124	0508		1174	0697		1224	0878		1274	1052	
1125	0512	4	1175	0700	3	1225	0881	3	1275	1055	3
1126	0515	3	1176	0704	4	1226	0885	4	1276	1059	4
		4			4			3			3
1127	0519		1177	0708		1227	0888		1277	1062	
1128	0523	4	1178	0711	3	1228	0892	4	1278	1065	3
1129	0527	4	1179	0715	4	1229	0896	4	1279	1069	4
		4			4			3			3
1130	0531		1180	0719		1230	0899		1280	1072	
1131	0535	4	1181	0722	3	1231	0903	4	1281	1075	3
1132	0538	3	1182	0726	4	1232	0906	3	1282	1079	4
1133	0542	4	1183	0730	4	1233	0910	4	1283	1082	3
		4			4			3			4
1134	0546		1184	0734		1234	0913		1284	1086	
1135	0550	4	1185	0737	3	1235	0917	4	1285	1089	3
1136	0554	4	1186	0741	4	1236	0920	3	1286	1092	3
		4			4			4			4
1137	0558		1187	0745		1237	0924		1287	1096	
1138	0561	3	1188	0748	3	1238	0927	3	1288	1099	3
1139	0565	4	1189	0752	4	1239	0931	4	1289	1103	4
1140	0569	4	1190	0755	3	1240	0934	3	1290	1106	3
1141	0573	4	1191	0759	4	1241	0938	4	1291	1109	3
1142	0577	4	1192	0763	4	1242	0941	3	1292	1113	4
1143	0580	3	1193	0766	3	1243	0945	4	1293	1116	3
		4			4			3			3
1144	0584		1194	0770		1244	0948		1294	1119	
1145	0588	4	1195	0774	4	1245	0952	4	1295	1123	4
1146	0592	4	1196	0777	3	1246	0955	3	1296	1126	3
		4			4			4			3
1147	0596		1197	0781		1247	0959		1297	1129	
1148	0599	3	1198	0785	4	1248	0962	3	1298	1133	4
1149	0603	4	1199	0788	3	1249	0966	4	1299	1136	3
1150	0607	4	1200	0792	4	1250	0969	3	1300	1139	3

pp.	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

pp.	3
1	0.3
2	0.6
3	0.9
4	1.2
5	1.5
6	1.8
7	2.1
8	2.4
9	2.7

CHAPTER XX

LOGARITHMS OF DECIMALS, POWERS AND ROOTS

149. Logarithms of Decimals.—While studying the preceding chapter, the student has probably been wondering about the value of the characteristic for the logarithm of a number that is entirely a decimal. According to Chapter XIX, the characteristic for numbers between 1 and 10 is 0; for numbers between 10 and 100 it is 1; for numbers between 100 and 1000 it is 2; etc. According to this system, we would expect the characteristic for decimals between 0 and 1 to be *less than* 0, or it would be negative (−). This is the exact truth of the matter, as we can demonstrate, by a simple calculation.

Let us find the logarithm of .234.

If this were 2.34, we would know that the characteristic is 0. From the table we find that the mantissa is .3692. Then we have

$$\log 2.34 = 0.3692$$

But

$$.234 = \frac{2.34}{10}$$

and since, in using logarithms, we divide by subtracting logarithms, we can get the log .234 by subtracting log 10 from log 2.34

$$\begin{aligned} \log 2.34 &= 0.3692 \\ \log 10 &= 1 \\ \log .234 &= 0.3692 - 1 \end{aligned}$$

In other words, we have a negative characteristic (−1) but the mantissa is still positive. This is sometimes written with the characteristic in the usual place, but with a − sign over it, thus $\bar{1}.3692$. We put the minus sign (−) over the 1 to indicate that only the characteristic is negative and that the mantissa is still positive, the logarithm being $-1+.3692$.

If the decimal had been .0234, we would have divided 2.34 by 100 and the characteristic of the log .0234 would have been $\bar{2}$. Similarly, the characteristic of log .00234 would be $\bar{3}$. From

these illustrations we derive the following rule: The characteristic of a decimal whose value is less than 1 is negative and is 1 more than the number of zeros immediately following the decimal point.

In order to avoid the confusion of having a logarithm which has negative and positive figures placed together, the following modification is generally made:

Instead of -1 we can write $9-10$
Instead of -2 we can write $8-10$, etc.

Then log .234 is written $9.3692-10$
log .0234 is written $8.3692-10$
log .00234 is written $7.3692-10$

Observe the following illustrations of logarithms of decimals:

$\log .000264 = 6.4216-10$
$\log .247 = 9.3927-10$
$\log .00446 = 7.6493-10$
$\log .000005 = 4.6990-10$

Notice that the difference between the characteristic and 10 is 1 more than the number of zeros immediately following the decimal point.

To understand the use of these in problems of multiplication or division we will work the following examples:

Example 1:

Multiply $327.6 \times .0729 \times .0028$.

log 327.6	$= 2.5153$
log .0729	$= 8.8627-10$
log .0028	$= 7.4472-10$
log of product	$= \overline{18}.8252-20$

This is the same as $8.8252-10$ or $\bar{2}.8252$
Antilog $8.8252-10 = .06687$, *Answer.*

The characteristic is $8-10$, or -2, which indicates that there should be one zero following the decimal point.

Example 2:

Divide 825 by .00872.

log 825	$= 2.9165$, or	$12.9165-10$
log .00872	$=$	$7.9405-10$
log of quotient	$=$	4.9760

Antilog $4.9760 = 94{,}620$, *Answer.*

In order to subtract the logs above, it is necessary to add $10-10$ to the 2.9165, making it $12.9165-10$.

Example 3:

Divide .000276 by 6930.

log .000276 $=6.4409-10$
log 6930 $=3.8407$
log of quotient $=2.6002-10$
Antilog $2.6002-10=.00000003983$, *Answer.*

Since the characteristic is $2-10$, or -8, there is 1 less, or 7 ciphers after the decimal point.

Example 4:

Divide 327.6 by .08274.

log 327.6	$=2.5135$	$=12.5153-10$
log .08274		$=\ 8.9177-10$
log of quotient		$=\ 3.5976$
Antilog 3.5976		$=3959$, *Answer.*

In order to subtract above, it was necessary first to add and subtract 10 from 2.5153, making it $12.5153-10$.

150. Powers and Roots.—Since the square of a number is the number multiplied by itself, it is self-evident that the logarithm of the square of a number is twice the log of the number; and that the log of the cube is three times the log of the number; etc. From these conclusions we have the following rule for obtaining powers by the use of logarithms:

Rule.—To raise a number to a power, we multiply its logarithm by the number of the power required and look up the antilogarithm.

Example 1:

Find value of 271^3.

log 271 $=2.4330$
$\times\ 3$
log of power $=7.2990$
Antilog $7.2990=19{,}900{,}000$, *Answer.*

Example 2:

Find value of $.000876^3$.

log .000876 $=6.9425-10$
$\times\ 3$
log of power $=20.8275-30$

In finding the antilog of this we find the number corresponding to the mantissa as usual. The characteristic is $20-30=-10$. Hence, there must be 9 ciphers following the decimal point.

Antilog $20.8275-30=.0000000006722$, *Answer.*

The square root of a number is one of the two equal factors of the number. Then, if we divide the logarithm of a number by 2 we will have the logarithm of one of the two equal factors of the number, or we will have the logarithm of its square root. Similarly, dividing the logarithm of a number by 3 will give the

logarithm of its cube root. This gives us the following rule for finding a root of any number:

Rule.—To extract the root of a number, we divide the logarithm of the number by the number of the root required, and look up the antilogarithm.

Example 1:

Find the square root of 324.9.

$$\log 324.9 = 2.5118$$
$$2)2.5118$$
$$1.2559$$
$$\text{antilog } 1.2559 = 18.03$$

Hence, $\sqrt{324.9} = 18.03$, *Answer.*

Example 2:

Find the diameter of a circle the area of which is 86 sq. in.

$$A = .7854\ D^2; \text{ or } D = \sqrt{\frac{A}{.7854}}$$
$$\log 86 = 1.9345 = 11.9345 - 10$$
$$\log .7854 = \underline{\quad 9.8945 - 10}$$
$$\log 86 \div .7854 = 2.0400$$
$$2)2.0400$$
$$1.0200$$
$$\text{antilog } 1.0200 = 10.472, \textit{ Answer.}$$

Example 3:

Find value of $\sqrt[3]{.0000732}$.

$$\log .0000732 = 5.8645 - 10$$

To get the cube root we should divide this log by 3, but a case like this requires special treatment, since as it now stands, it would give a rather complicated result. Let us take the logarithm, $5.8645 - 10$, and arrange it so the last term (-10) will be divisible by the number of the root desired (3). The simplest way is to make this term 30. The logarithm will then be $25.8645 - 30$.

We can now proceed.

$$3)25.8645 - 30$$
$$8.6215 - 10$$
$$\text{antilog } 8.6215 - 10 = .04183$$

Therefore, $\sqrt[3]{.0000732} = .04183$, *Answer.*

151. Fractional Powers.—There are sometimes relations between quantities that can be expressed only by using a decimal exponent with one quantity. Such equations are readily solved by the use of logarithms. For example, in calculating the compression pressure in a gasoline engine we have the relation:

$$P_2 = P_1\, r^{1.28}$$

This equation means that the pressure P_2 after compression equals the pressure P_1 in the cylinder before compression times the compression ratio raised to the 1.28 power. It would be out

of the question to attempt to solve such an equation by the ordinary processes of arithmetic but it is quite easily done by the use of logarithms. To take a concrete case, let us suppose that the pressure in a cylinder is 14 lb. per square inch before compression and that the gas is compressed 5 times; that is, it is to be compressed into a space one-fifth of that which it occupies before compression. Then P_1 is 14 and r is 5. The equation for P_2 is, then,

$$P_2 = 14 \times 5^{1.28}$$

We can get the logarithm of $5^{1.28}$ by taking the log of 5 and multiplying it by 1.28. To this we can add the log of 14. The sum will be the log of P_2.

$$\begin{aligned} \log 5 &= 0.6990 \\ \log 5^{1.28} &= 0.6990 \times 1.28 = 0.8947 \\ \log 14 &= \quad\quad 1.1461 \\ \log P_2 &= \quad\quad 2.0408 \end{aligned}$$

$P_2 =$ antilog $2.0408 = 109.8+$ lb. per square inch, *Answer.*

152. Logarithms of Trigonometric Functions.—In practically every problem where we use a function of an angle, the processes used are those of multiplication or division. Usually we multiply some function by one side of a triangle to get some other side. In many such cases time will be saved by using logarithms. To avoid the necessity of looking in one table for the function of an angle and then referring to a table of logarithms to find the log of that function, many handbooks contain tables which give the logarithm of the function direct. Such tables are usually labelled "Logarithmic Functions of Angles" or "Logarithms of Trigonometric Functions," while the tables giving the real functions are labelled "Natural Functions." For this reason the table given at the end of Chapter XIV is headed "Natural Functions of Angles." In using any handbook, the student should use particular care to observe whether the table of functions of angles is a natural table or a logarithmic table.

PROBLEMS

Note.—Use logarithms in all calculations.

201. Find the logarithms of the following numbers: (a) .736; (b) .00829; (c) .003216; (d) 4.217; (e) .0000429; (f) .2719; (g) .00000000981.

202. Find the antilogarithms of the following logarithms: (a) $9.8216-10$; (b) $6.2704-10$; (c) $7.0819-10$; (d) $4.3074-10$; (e) $8.3240-20$; (f) 0.2719.

203. Find the value of $\sqrt[3]{86400}$.

204. Find the value of 4.375^4.

205. Find the value of $.2796^2$.

206. Find the square root of .07284.

207. The volume of a sphere is

$$V = \frac{\pi}{6} D^3$$

where D is the diameter. Solve the formula for D and find the diameter of a ball containing 92.5 cu. in.

208. The following formula is commonly used to get the horse-power that can safely be transmitted by wrought iron or steel line-shafting having bearings not over 8 ft. apart:

$$H = \frac{d^3 R}{90}$$

H is the horse-power; d is the diameter of the shaft in inches; and R is the revolutions per minute.

Find the horse-power that can be transmitted by a $2\frac{1}{2}$-in. shaft running at 225 R. P. M.

209. Transpose the above formula so as to find d when the horse-power and R. P. M. are given; and then find the diameter of shaft necessary to transmit 25 H. P. at 200 R. P. M.

210. The safe working pressure for a boiler or other cylindrical shell is obtained from the following formula:

$PD = 2\ TSE$

P = pressure in pounds per square inch
D = diameter of the shell in inches
T = thickness of the plate from which shell is made
S = allowable stress per square inch in the plate
E = efficiency of the riveted joint in the plate.

Solve this formula for P and then, using logarithms, find the safe working pressure for a boiler 60 in. in diameter, with plate $\frac{3}{8}$ in. thick. Allowable stress in plate = 12,000 lb. per square inch. Efficiency of joint = .75.

211. In calculating the size of a circular pipe to deliver fresh air for ventilating a shop the following formula is frequently used:

$$d = \sqrt[5]{\frac{Q^2 L}{10.8\ p}}$$

where, d is the diameter of the pipe in inches; Q is the number of cubic feet of air to be discharged per second; L is the distance in feet from the fan to the outlet; and p is the pressure at fan in pounds per square inch.

Find the diameter of pipe to deliver 30 cu. ft. of air per second through a pipe 125 ft. long with a pressure at the fan of .01 lb. per square inch.

212. Calculate the compression pressure in a gasoline engine when the initial pressure before compression is 14 lb. and the compression ratio is 4.5.

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